Representable and Diagonally Representable Weakening Relation Algebras

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Section 1

Introduction

Proper Relation Algebras

A *proper relation algebra* is a structure $\mathcal{A} = (A, \bot, \top, \neg, +, ^{\smile}, 1, ;)$ where

- $A \subseteq \wp(X \times X)$
- $\bot, \top, +, \neg$ are interpreted as Boolean bottom, top, join (union), complementation ($\neg R = \top \setminus R$)
- 1 is interpreted as the diagonal relation over X and it is the identity for ; defined as

$$R;S = \{(x, y) \mid \exists z : (x, z) \in R, (z, y) \in S\}$$

- $^{\smile}$ is interpreted as relational converse

$$R^{\smile} = \{(y, x) \mid (x, y) \in R\}$$

for $R, S \subseteq X \times X$
(We can also define meet $R \cdot S = \neg(\neg R + \neg S)$ and $0 = \neg 1$)

Weakening Relations

Let $\mathbf{X} = (X, \leq)$ be a poset. A relation $R \subseteq X \times X$ is a *weakening relation* if and only if

$$\leq;R = R;\leq \, = \, \leq;R;\leq \, = R$$

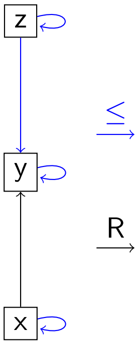
Operations in RA for Weakening Relations

Weakening relations are closed under

- Boolean $\bot, \top$
- Join $+$
- Composition ;

but not

- Complement $\neg$
- Converse $\smile$



Complement-Converse to the Rescue

Define a unary operation $\sim$ for some $R \subseteq X \times X$ as

$$\sim R = \neg(R^{\smile}) = (\neg R)^{\smile}$$

or explicitly

$$\sim R = \{(x, y) \mid (y, x) \in \top \setminus R\}$$

Interestingly, weakening relations are closed under $\sim$

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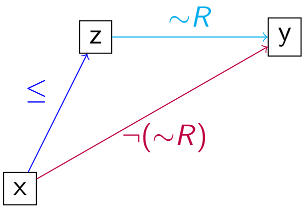

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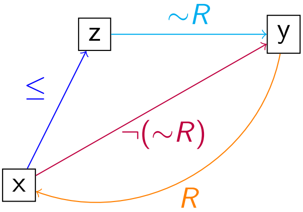

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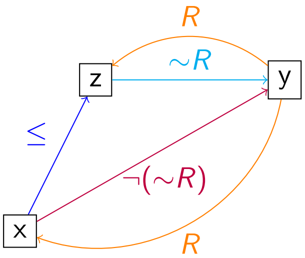


Proper Weakening Relation Algebras

A *proper weakening relation algebra* is a structure $\mathcal{A} = (A, \bot, \top, +, \sim, 1, ;)$ where

- $A \subseteq \wp(X \times X)$
- $(X, 1)$ is a poset
- 1 is a two-sided identity for ;
- $\bot, \top, +$ are interpreted as Boolean bottom, top, join (union)
- $\sim, ;$ are interpreted as relational complement-converse, composition

(We can also define meet $R \cdot S = \sim(\sim R + \sim S)$ and $0 = \sim 1$)

Representable Weakening Relation Algebras

- The class of *representable weakening relation algebras RwkRA* is the class of all $(A, \perp, \top, +, \sim, 1, ;)$-structures $\mathcal{A}$ isomorphic to a proper $\mathcal{A}'$ via some isomorphism h
- h is called a *representation*
- If for $\mathcal{A}$ there exists a representation h' such that $h'(1)$ is an antichain, then $\mathcal{A}$ is *diagonally representable*

Section 2

Theory of RwkRA

Result Summary

Proposition. Membership in *RwkRA* is undecidable for finite structures.

Corollary. *RwkRA* is not finitely axiomatisable.

Proposition. *RwkRA* is not closed under homomorphisms.

Corollary. *RwkRA* is not a variety.

Theorem. Every simple diagonally representable weakening relation algebra has a discriminator term.

Corollary. The class *DRwkRA* is a discriminator variety.

Non Finite Axiomatisability

Lemma. For every relation algebra, its $(\perp, \top, +, \sim, 1, ;)$-reduct can only be diagonally representable.

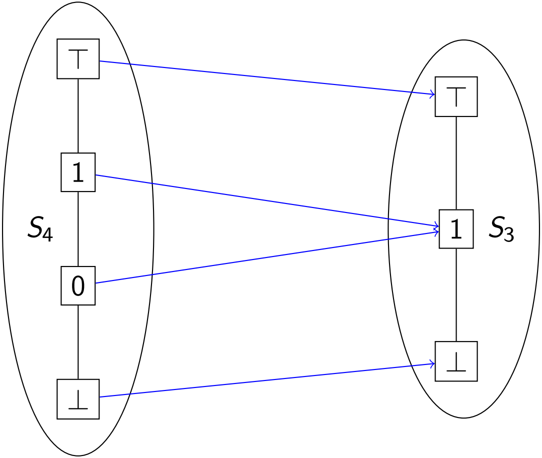
Theorem. A relation algebra is representable if and only if its $(\perp, \top, +, \sim, 1, ;)$-reduct is.

Corollary. Membership in *RwkRA* is undecidable for finite structures.

Corollary. *RwkRA* is not finitely axiomatisable.

RwkRA is not a Variety

Even-length Sugihara monoids are in *RwkRA*, but odd-length Sugihara monoids are not in *RwkRA*.



RwkRA is not closed under homomorphisms.

DRwkRA is a Discriminator Variety

A *discriminator term* d for a simple algebra is a term defined for three

$$d(a, b, c) = \begin{cases} c & \text{if } a = b \\ a & \text{otherwise} \end{cases}$$

If you have a term $t(a, b)$ that evaluates to $\bot$ if and only if $a = b$ then the discriminator term for simple representable [weakening] relation algebras can be defined as

$$(\top;t(a,b);\top \cdot a) + (\neg(\top;t(a,b);\top) \cdot c)$$
$$(\top;t(a,b);\top \cdot a) + ({\sim}(\top;t(a,b);\top) \cdot c)$$

For BA, $t(a, b)$ is defined as $(\neg a \cdot b) + (\neg b \cdot a)$

DRwkRA is a Discriminator Variety

We define a term $t(a, b)$ for *DRwkRA* as

$$1 \cdot (a;(b \cdot {\sim}b)) + 1 \cdot ({\sim}b;(a \cdot {\sim}b)) + 1 \cdot (b;(a \cdot {\sim}a)) + 1 \cdot ({\sim}a;(b \cdot {\sim}a))$$

and we have

Theorem. Every simple diagonally representable weakening relation algebra has a discriminator term.

Corollary. The class *DRwkRA* is a discriminator variety.

Section 3

Towards the Abstract Class of Weakening Relation Algebras

Abstract Relation Algebras

An *(Abstract) Relation Algebra* is an algebra $(A, \bot, \top, \neg, +, 1, ^{\smile}, ;)$ such that

1. its $(A, \bot, \top, \neg, +)$-reduct is a Boolean Algebra
2. 1 is the identity for associative and additive ;
3. $(a^{\smile})^{\smile} = a$, $(a; b)^{\smile} = (b^{\smile});(a^{\smile})$, and $^{\smile}$ is additive
4. the DeMorgan-Tarski equation holds: $(a^{\smile});\neg(a;b) \leq \neg b$

Abstract Classes of RA

- for $2 \leq n \leq \omega$, RA_n, the class of all relation algebras that
  - have an n-dimensional base
  - can be characterised using n variables in FOL
  - 'the builder' has a winning strategy for the n-pebble version of a representation game
- $RA_4 = RA$
- $RA_\omega = RRA$

$$
\begin{array}{c}
RA_2 \\
\cup\!\!| \\
RA_3 \\
\cup\!\!| \\
RA = RA_4 \\
\cup\!\!| \\
\vdots \\
\cup\!\!| \\
RRA = RA_\omega
\end{array}
$$

Axioms for $wkRA_2$

1. Axioms for bounded cyclic involutive unital $d\ell$-magmas
2. $s \cdot {\sim}s \leq 0$
3. $s \leq t \Leftrightarrow s; {\sim}t \cdot 1 \leq 0$
4. $s \leq t;u \wedge s;t \leq {\sim}u \Rightarrow s \cdot 1 \leq 0$
5. $s \leq t;u \wedge u;s \leq {\sim}t \Rightarrow s \cdot 1 \leq 0$
6. $s \leq t;u \wedge (s \cdot 1 \cdot t;v) + (1 \cdot s \cdot {\sim}v;u) \leq 0 \Rightarrow s \cdot 1 \leq 0$

Axioms for $wkRA_3$

1. Axioms for $wkRA_2$
2. $s; t \leq \sim u \Rightarrow t; u \leq \sim s$
3. $s \cdot t;u \leq ((s;v) \cdot t);u + t;(u \cdot \sim v)$
4. $1 \cdot \sim s';s \cdot t;\sim t' \leq 0 \Rightarrow s;t \leq (s \cdot s');t + s;(t \cdot t')$
5. $1 \cdot s \cdot 0 = \bot \Rightarrow (s \cdot 1);(t;u) \leq ((s \cdot 1);t);u$
6. $1 \cdot u \cdot 0 = \bot \Rightarrow (s;t);(u \cdot 1) \leq s;(t;(u \cdot 1))$

Axioms for *wkRA*

Axioms for RA_3 with associativity axiomatise RA_4. Is the same true for $wkRA_4$?

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Axioms for RA_3 with associativity axiomatise RA_4. Is the same true for $wkRA_4$? — It remains open.

Is $wkRA_4$ finitely axiomatisable?

$RA_i, 4 < i$ are not finitely axiomatisable, does the same hold for $wkRA_i$?

Some other things in the paper

1. Presenting weakening relation algebras as relevance frames
2. Frame axiomnatisations of the classes $wkRA_2, wkRA_3$
3. Weakening relation algebra by games (for algebras and frames)
4. Explicit representations of all associative members of $wkRA_3$ up to size 6

Acknowledgment

The authors gratefully acknowledge a very useful conversation with Roger Maddux regarding relevance frames, relevance logic and its connections with relation algebras.