Dependences between Domain Constructions in Heterogeneous Relation Algebras

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1. Heterogeneous Relation Algebras
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4. Independences

Heterogeneous Relation Algebras

heterogeneous relation

- $R : A \leftrightarrow B$ for $A = \{1, 2\}$ and $B = \{a, b, c\}$
- $R = \{(1, b), (1, c), (2, a), (2, c)\} \subseteq A \times B$

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

locally small category $(\mathrm{Obj}, \mathrm{Mor}(A, B), ;, \mathsf{I}_A)$ with

- complete atomic Boolean algebra $(\mathrm{Mor}(A, B), \sqcup_{A,B}, \sqcap_{A,B}, \overline{}_{A,B}, \mathsf{O}_{A,B}, \mathsf{L}_{A,B}, \sqsubseteq_{A,B})$
- transposition ${}^{\mathsf{T}}_{A,B} : \mathrm{Mor}(A, B) \to \mathrm{Mor}(B, A)$
- Schröder equivalences $QR \sqsubseteq S \Leftrightarrow Q^{\mathsf{T}}\overline{S} \sqsubseteq \overline{R} \Leftrightarrow \overline{S}R^{\mathsf{T}} \sqsubseteq \overline{Q}$
- Tarski rule $R \neq \mathsf{O} \Leftrightarrow \mathsf{L}R\mathsf{L} = \mathsf{L}$

REL: all binary relations between non-empty sets

Mappings

R is

- injective if $RR^{\mathsf{T}} \sqsubseteq \mathsf{I}$
- total if $\mathsf{I} \sqsubseteq RR^{\mathsf{T}}$
- univalent if R^{T} is injective
- a mapping if R is total and univalent

REL: R injective if $\forall x, y, z : (x, z) \in R \wedge (y, z) \in R \Rightarrow x = y$
R total if $\forall x : \exists y : (x, y) \in R$

Domain Constructions

- power sets
- products
- sums
- quotients
- subsets

Power Sets

symmetric quotient

- $Q \div R = (Q \backslash R) \sqcap (R \backslash Q)^{\mathsf{T}}$
- right residual $Q \backslash R = \overline{Q^{\mathsf{T}} \overline{R}}$

power of object A

- object 2^A
- membership relation $\varepsilon : A \leftrightarrow 2^A$
- $\varepsilon \div \varepsilon \sqsubseteq \mathsf{I}$
- $R \div \varepsilon$ total for each R

REL: powerset, $(x, Y) \in \varepsilon \Leftrightarrow x \in Y$

Products

product of objects A and B

- object $A \times B$
- projections $p_A : A \times B \leftrightarrow A$ and $p_B : A \times B \leftrightarrow B$
- p_A and p_B mappings
- ${p_A}^\mathsf{T} p_B = \mathsf{L}$
- $p_A {p_A}^\mathsf{T} \sqcap p_B {p_B}^\mathsf{T} \sqsubseteq \mathsf{I}$

REL: Cartesian product, $((x, y), z) \in p_A \Leftrightarrow x = z$

Sums

sum of objects A and B

- object $A + B$
- injections $i_A : A \leftrightarrow A + B$ and $i_B : B \leftrightarrow A + B$
- i_A and i_B injective mappings
- $i_A {i_B}^\mathsf{T} = \mathsf{O}$
- $\mathsf{I} \sqsubseteq {i_A}^\mathsf{T} i_A \sqcup {i_B}^\mathsf{T} i_B$

REL: disjoint union, $(x, (y, Z)) \in i_A \Leftrightarrow x = y \wedge Z = A$

Quotients

equivalence E

- reflexive $\mathsf{I} \sqsubseteq E$
- symmetric $E^\mathsf{T} = E$
- transitive $EE \sqsubseteq E$

quotient of object A by equivalence E

- object A/E
- projection $p : A \leftrightarrow A/E$
- $pp^\mathsf{T} = E$
- $p^\mathsf{T} p = \mathsf{I}$

REL: equivalence classes, $(x, Y) \in p \Leftrightarrow Y = \{y \mid (x, y) \in E\}$

Subsets

partial identity S

- $S \sqsubseteq \mathsf{I}$

subset of object A corresponding to partial identity $S \neq \mathsf{O}$

- object S
- injection $i : S \leftrightarrow A$
- $i^\mathsf{T} i = S$
- $i i^\mathsf{T} = \mathsf{I}$

REL: subset, $(x, y) \in i \Leftrightarrow x = y \wedge (x, x) \in S$

Research

- axioms characterise domains uniquely up to isomorphism
- study (in)dependence of axioms

results

- Assume all power sets and subsets exist and objects are comparable.
  Then all sums exist.
- Assume all sums exist and atoms are rectangular.
  Then all products exist.
- Assume all atoms are rectangular.
  Then all subsets exist if and only if all quotients exist.
- Assume all atoms are rectangular.
  Then there are no further dependences.

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Sums from Power Sets and Subsets

Assume all subsets and power sets exist.
Then $A + A$ exists for each object A.

- for $A = \{1, 2\}$ construct $\{\{\{1\}\}, \{\{2\}\}, \{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}\}$
- injections $i_A, i_B : A \leftrightarrow 2^{2^A}$ with $i_A = (\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon)$ and $i_B = (\varepsilon \backslash \mathsf{I}) \div \varepsilon$
- take subset corresponding to range of injections

Assume all subsets and power sets exist and objects are comparable.
Then $A + B$ exists for each object A, B.

- B contained in A if there is an injective mapping $i : B \leftrightarrow A$
- A, B comparable if B contained in A or A contained in B
- $i_B = i((\varepsilon \backslash \mathsf{I}) \div \varepsilon) : B \leftrightarrow 2^{2^A}$ injects B into A, then into $A + A$

Products from Power Sets and Subsets

Assume all atoms are rectangular.
Then all objects are comparable.

- Q atom if $Q \neq \mathsf{O}$ and, for each $R \sqsubseteq Q$, either $R = Q$ or $R = \mathsf{O}$
- R is rectangular if $R\mathsf{L}R \sqsubseteq R$

Assume all subsets and power sets exist and atoms are rectangular.
Then $A \times B$ exists for each object A, B.

- for $A = \{1, 2\}$ and $B = \{a, b, c\}$ construct $\{\{1, a\}, \{1, b\}, \{1, c\}, \{2, a\}, \{2, b\}, \{2, c\}\}$
- construct $A + B$ by previous theorem
- projections $p_A = i_A \varepsilon \div \mathsf{I} : 2^{A+B} \leftrightarrow A$ and $p_B = i_B \varepsilon \div \mathsf{I} : 2^{A+B} \leftrightarrow B$

Products from Sums

Assume all sums exist and atoms are rectangular.
Then $A \times B$ exists for each object A, B.

finitely many atomic partial identities $\mathrm{at}_1(B) = \{b_1, \ldots, b_n\}$

- $A \times B = A_n = A + \cdots + A$ (n summands)
- $A_1 = A$ and $A_k = A_{k-1} + A$ with $i_k : A_{k-1} \leftrightarrow A_k$ and $j_k : A \leftrightarrow A_k$
- compose projections from injections

$\mathrm{at}_1(B)$ infinite

- $A \times B = A$ if $|\mathrm{at}_1(A)| \geq |\mathrm{at}_1(B)|$, otherwise $A \times B = B$
- bijection between infinite sets of atomic partial identies by Cantor-Schröder-Bernstein

Subsets from Quotients and Vice Versa

Assume all atoms are rectangular.
Then all quotients exist if and only if all subsets exist.

subsets from quotients

- partial identity $S : A \leftrightarrow A$ with atom $a \sqsubseteq S$
- equivalence $E = S \sqcup a\mathsf{L}\neg S \sqcup \neg S\mathsf{L}a \sqcup \neg S\mathsf{L}\neg S$ with $\neg S = \overline{S\mathsf{L}} \sqcap \mathsf{I}$
- A/E is subset corresponding to S

quotients from subsets

- equivalence $E : A \leftrightarrow A$
- equivalence $\sim$ on $\mathrm{at}_1(A)$ by $a \sim b \Leftrightarrow a\mathsf{L}b \sqsubseteq E$
- partial identity $S = \bigsqcup_{i \in I} a_i$ for representatives a_i of $\mathrm{at}_1(A)/{\sim}$
- subset corresponding to S is A/E

Independences

power	product	sum	subset	objects
no	no	no	no	$\mathbf{2}$
no	no	no	yes	$\mathbf{1}, \mathbf{2}$
no	no	yes	no	no model
no	no	yes	yes	no model
no	yes	no	no	$\mathbf{1}, \mathbb{N}$
no	yes	no	yes	$\mathbf{1}$
no	yes	yes	no	$\mathbb{N}$
no	yes	yes	yes	$\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots, \mathbb{N}$
yes	no	no	no	$\mathbf{2^i}, \mathbf{3^i}$ for $i \in \mathbb{N}$
yes	no	no	yes	no model
yes	no	yes	no	no model
yes	no	yes	yes	no model
yes	yes	no	no	$\mathbf{2^i}$ for $i \in \mathbb{N}$
yes	yes	no	yes	no model
yes	yes	yes	no	$\mathbf{2}, \mathbf{3}, \mathbf{4}, \dots$
yes	yes	yes	yes	$\mathbf{1}, \mathbf{2}, \mathbf{3}, \dots$

subalgebras of REL: **k** is *k*-element set, all morphisms

Conclusion

- weaken assumptions of comparability and rectangular atoms?
- weaker relational products
- allegories, Dedekind categories