Relational Algebraic Approach to the Real Numbers
The Additive Group

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Motivation

1. Categories of relations have been used to present mathematical theories in an equational style; including program semantics.
2. In these applications certain objects such as abstract versions of singletons and natural numbers as well as constructions on objects such as relational sums, products, powers are essential.

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5. We will utilize Tarski's axioms of the real numbers because of their simplicity and compactness.
6. We will formulate the requirements in a negation-free style so that these results also transfer to so-called L-fuzzy relations.

Heyting Categories

Definition

A Heyting category $\mathcal{R}$ is a category satisfying the following:

1. For all objects A and B the collection $\mathcal{R}[A, B]$ is a Heyting algebra. Meet, join, relative pseudo-complement, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \rightarrow, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$, respectively.
2. $Q; \perp\!\!\!\perp_{BC} = \perp\!\!\!\perp_{AC}$ for all relations $Q : A \rightarrow B$.
3. There is a monotone operation $\smile$ (called converse) mapping a relation $Q : A \rightarrow B$ to $Q^{\smile} : B \rightarrow A$ such that for all relations $Q : A \rightarrow B$ and $R : B \rightarrow C$ the following holds: $(Q; R)^{\smile} = R^{\smile}; Q^{\smile}$ and $(Q^{\smile})^{\smile} = Q$.
4. For all relations $Q : A \rightarrow B, R : B \rightarrow C$ and $S : A \rightarrow C$ the modular inclusion $(Q; R) \sqcap S \sqsubseteq Q; (R \sqcap (Q^{\smile}; S))$ holds.
5. For all relations $R : B \rightarrow C$ and $S : A \rightarrow C$ there is a relation $S/R : A \rightarrow B$ (called the right residual of S and R) such that for all $X : A \rightarrow B$ the following holds: $X; R \sqsubseteq S \iff X \sqsubseteq S/R$.

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Considering concrete relation a map $p : 1 \to A$, i.e., a relation that maps the only element $*$ of 1 to one element a in A, can be identified with the element a. Therefore we call a map $p : 1 \to A$ a point (of A).

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Definition

An object $A \times B$ together with two relations $\pi : A \times B \to A$ and $\rho : A \times B \to B$ is called a relational product iff

$$\pi^{\smile};\pi \sqsubseteq \mathbb{I}_A, \quad \rho^{\smile};\rho \sqsubseteq \mathbb{I}_B, \quad \pi;\pi^{\smile} \sqcap \rho;\rho^{\smile} = \mathbb{I}_{A\times B}, \quad \pi^{\smile};\rho = \mathbb{T}_{AB}.$$

Relational Products

Given relational products we will use the following abbreviations

$$Q \olessthan R = Q;\pi^\smile \sqcap R;\rho^\smile,$$
$$Q \ogreaterthan S = \pi;Q \sqcap \rho;S,$$
$$Q \otimes T = \pi;Q;\pi^\smile \sqcap \rho;T;\rho^\smile = Q;\pi^\smile \ogreaterthan T;\rho^\smile = \pi;Q \olessthan \rho;T.$$

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For concrete relation $Q \olessthan R$ is given by

$$(Q \olessthan R)(x,(y,z)) \text{ iff } Q(x,y) \text{ and } R(x,z).$$

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We also use the following two bijective mappings $\mathrm{assoc} : A \times (B \times C) \to (A \times B) \times C$ and $\mathrm{swap} : A \times B \to B \times A$ defined by

$$
\begin{aligned}
\mathrm{assoc} &= \pi;\pi^{\smile};\pi^{\smile} \sqcap \rho;\pi;\rho^{\smile};\pi^{\smile} \sqcap \rho;\rho;\rho^{\smile} = (\mathbb{I}_A \otimes \pi) \olessthan \rho;\rho = \pi^{\smile};\pi^{\smile} \ogreaterthan (\rho^{\smile} \otimes \mathbb{I}_C), \\
\mathrm{swap} &= \pi;\rho^{\smile} \sqcap \rho;\pi^{\smile} = \rho \olessthan \pi = \rho^{\smile} \ogreaterthan \pi^{\smile}.
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Relational Powers

Definition

An object $\mathcal{P}(A)$ together with a relation $\varepsilon : A \to \mathcal{P}(A)$ is called a relational (or direct) power of A iff

$$\mathrm{syQ}(\varepsilon, \varepsilon) = \mathbb{I}_{\mathcal{P}(A)} \quad \text{and} \quad \mathrm{syQ}(Q, \varepsilon) \text{ is total for every } Q : A \to B.$$

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The relation $\mathrm{syQ}(R^{\smile}, \varepsilon)$ is a map for every relation $R : B \to A$. For concrete relations, this construction yields the existential image of R, i.e.,

$$\mathrm{syQ}(R^{\smile}, \varepsilon)(x, M) \text{ iff } M = \{y \mid (x, y) \in R\}.$$

Internal Abelian Groups

Definition

A quadruple (A, e, f, n) in a Heyting category $\mathcal{R}$ is called an abelian group iff A is an object, $e : 1 \to A$ is a point, and $f : A \times A \to A$ and $n : A \to A$ are maps satisfying:

1. f is associative, i.e., $(\mathbb{I}_A \otimes f);f = \text{assoc};(f \otimes \mathbb{I}_A);f$,
2. e is the neutral element of f, i.e., $(\mathbb{I}_A \oslash \pi_{A1};e);f = \mathbb{I}_A$,
3. n is the complement map, i.e., $(\mathbb{I}_A \oslash n);f = \pi_{A1};e$,
4. f is commutative. i.e., $\text{swap};f = f$.

Internal Linearly Ordered Abelian Groups

Definition

A relation $C : X \to X$ is called

1. transitive if $C;C \sqsubseteq C$,
2. dense if $C \sqsubseteq C;C$,
3. asymmetric if $C \sqcap C^{\smile} = \perp\!\!\!\perp_{XX}$,
4. a strict-order if C is transitive and asymmetric,
5. a linear strict-order if C is a strict-order and $\mathbb{I}_X \sqcup C \sqcup C^{\smile} = \top\!\!\!\top_{XX}$.

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Definition

A quintuple (A, e, f, n, C) in a Heyting category $\mathcal{R}$ is called a densely linearly ordered abelian group if (A, e, f, n) is an abelian group, C is a dense linear strict-order, and f is monotone with respect to the order $E = C \sqcup \mathbb{I}_A$, i.e., $(E \otimes E);f \sqsubseteq f;E$.

Tarski's Axioms of the Real Numbers

The following list constitute Tarski's axioms of the real numbers. His axioms are based on the language $\mathbb{R}, <, +, 1$:

Axiom 1: If $x \neq y$, then $x < y$ or $y < x$.

Axiom 2: If $x < y$, then $y \nless x$.

Axiom 3: If $x < z$, then there is a y such that $x < y$ and $y < z$.

Axiom 4: If $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ so that for every $x \in X$ and every $y \in Y$ we have $x < y$, then there is a z so that for all $x \in X$ and $y \in Y$ we have $x \leq z$ and $z \leq y$ ($x \leq y$ shorthand for $x < y$ or $x = y$).

Axiom 5: $x + (y + z) = (x + z) + y$.

Axiom 6: For every x and y there is a z such that $x = y + z$.

Axiom 7: If $x + z < y + t$, then $x < y$ or $z < t$.

Axiom 8: $1 \in \mathbb{R}$.

Axiom 9: $1 < 1 + 1$.

Real Number Object

Definition

An object $\mathbb{R}$ together with three relations $\imath : 1 \to \mathbb{R}$, $C : \mathbb{R} \to \mathbb{R}$ and $\mathrm{add} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called a real number object if the following holds:

0. add is a map.
1. $\mathbb{I}_{\mathbb{R}} \sqcup C \sqcup C^{\smile} = \top\!\!\top_{\mathbb{R}\mathbb{R}}$.
2. $C \sqcap C^{\smile} = \bot\!\!\bot_{\mathbb{R}\mathbb{R}}$.
3. $C \sqsubseteq C;C$.
4. $\varepsilon\backslash(C/\varepsilon^{\smile}) \sqsubseteq (\varepsilon\backslash(C \sqcup \mathbb{I}_{\mathbb{R}}));(\varepsilon\backslash(C \sqcup \mathbb{I}_{\mathbb{R}})^{\smile})^{\smile}$.
5. $(\mathbb{I}_{\mathbb{R}} \otimes \mathrm{add});\mathrm{add} = (\mathbb{I}_{\mathbb{R}} \otimes \mathrm{swap});\mathrm{assoc};(\mathrm{add} \otimes \mathbb{I}_{\mathbb{R}});\mathrm{add}$.
6. $\pi^{\smile};\mathrm{add} = \top\!\!\top_{\mathbb{R}\mathbb{R}}$.
7. $\mathrm{add};C;\mathrm{add}^{\smile} \sqsubseteq \pi;C;\pi^{\smile} \sqcup \rho;C;\rho^{\smile}$.
8. $\imath$ is a map, i.e., a point.
9. $\imath \sqsubseteq \imath;(\mathbb{I}_{\mathbb{R}} \obslash \mathbb{I}_{\mathbb{R}});\mathrm{add};C^{\smile}$.

Equivalence of the Axioms (Example)

Axiom 6

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&\iff \quad \forall x, y: \; \exists (u, z): \; \pi^{\smile}(y, (u, z)) \wedge \mathrm{add}((u, z), x)
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&\iff \quad \forall x,y:\ \exists(u,z):\ \pi^{\smile}(y,(u,z)) \wedge \mathrm{add}((u,z),x)\\
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$$\iff \quad \forall x,y,z,t:\ C(x+z, y+t) \Rightarrow (\pi; C; \pi^{\smile})((x,z),(y,t)) \vee (\rho; C; \rho^{\smile})((x,z),(y,t))$$

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Definition of 0 and neg

We define $Z, \mathrm{neg} : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} Z &= (\mathrm{add}^{\smile} \sqcap \pi^{\smile}); \rho, \\ \mathrm{neg} &= \pi^{\smile}; (\mathrm{add}; Z^{\smile} \sqcap \rho). \end{aligned}$$

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so that we define $0 = \pi_{1\mathbb{R}}; Z$.

Additive Abelian Group of $\mathcal{R}$

Lemma

1. $\mathrm{swap}; \mathrm{add} = \mathrm{add}$.
2. $(\mathbb{I}_{\mathbb{R}} \otimes \mathrm{add}); \mathrm{add} = \mathrm{assoc}; (\mathrm{add} \otimes \mathbb{I}_{\mathbb{R}}); \mathrm{add}$.
3. $\mathbb{T}_{\mathbb{R}\mathbb{R}}; Z = Z$.
4. $(\mathbb{I}_{\mathbb{R}} \otimes Z); \mathrm{add} = \mathbb{I}_{\mathbb{R}}$.
5. $\mathbb{I}_{\mathbb{R}} \otimes Z \sqsubseteq \mathrm{add}^{\smile}$.
6. $\mathrm{neg}^{\smile} = \mathrm{neg}$.

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3. $\mathbb{T}_{\mathbb{R}\mathbb{R}}; Z = Z$.
4. $(\mathbb{I}_{\mathbb{R}} \oslash Z); \mathrm{add} = \mathbb{I}_{\mathbb{R}}$.
5. $\mathbb{I}_{\mathbb{R}} \oslash Z \sqsubseteq \mathrm{add}^{\smile}$.
6. $\mathrm{neg}^{\smile} = \mathrm{neg}$.

Theorem

The quadruple $(\mathbb{R}, 0, \mathrm{add}, \mathrm{neg})$ *is an abelian group.*

Order Structure on $\mathcal{R}$

Lemma

1. $\mathrm{add};\mathrm{add}^{\smile} \sqcap \rho;\rho^{\smile} = \mathbb{I}_{\mathbb{R}\otimes\mathbb{R}}$.
2. $\mathrm{add};C;\mathrm{add}^{\smile} \sqcap \rho;\rho^{\smile} = C \otimes \mathbb{I}_{\mathbb{R}}$.

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The relation $C : \mathbb{R} \to \mathbb{R}$ is a dense linear strict-order.

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The quintuple $(\mathbb{R}, 0, \mathrm{add}, \mathrm{neg}, C)$ is a densely linearly ordered abelian group.

Conclusion and Future Work

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4. Topological features of real number objects using the relational theory of topological spaces should be investigated.
5. . . .