Poset Product Representations Over Simple Residuated Lattices

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Residuated lattices

Definition:

A (bounded, commutative, integral) residuated lattice is an algebra $(A, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ such that

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
- $(A, \cdot, 1)$ is a commutative monoid;
- For all $x, y, z \in A$;

$$x \cdot y \leq z \iff x \leq y \rightarrow z.$$

Recall that an algebraic structure is simple if it has no non-trivial congruences (quotients).

Familiar residuated lattices

Residuated lattices form an arithmetical (= congruence distributive + congruence permutable) variety with the congruence extension property. Examples include:

- Heyting algebras (where $\cdot$ is $\wedge$) and Boolean algebras.
- MTL-algebras, the algebraic semantics of t-norm based logics, satisfying $(x \to y) \vee (y \to x) = 1$ (residuated lattices that are subdirect products of totally ordered ones).
- GBL-algebras, satisfying divisibility $x(x \to y) = x \wedge y$.
- BL-algebras, the algebraic semantics of Petr Hájek's basic fuzzy logic, the intersection of MTL and GBL.
- MV-algebras, the algebraic semantics of Łukasiewicz logic, BL-algebras that satisfy $(x \to 0) \to 0 = x$.
- Gödel algebras, the algebraic semantics of Gödel-Dummett logic, the intersection of MTL and Heyting algebras.

Representations and their benefits

Representations of algebraic structures as algebras whose members are functions have a long history (e.g., representations of groups as groups of permutations). This talk is about representations of residuated lattices in terms of antichain labelings (special monotone functions). Benefits:

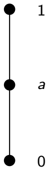
- Relational semantics for corresponding logics (F. 2022).
- Translations into natural modal logics (F.-Zuluaga Botero 2021).
- Decidability of the universal theory (Jipsen-Montagna 2010).
- Amalgamation and interpolation for the corresponding logics (Metcalfe-Montagna-Tsinakis 2014).

Antichain labelings

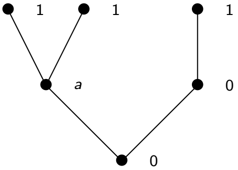
Definition:

Let $(X, \leq)$ be a poset, and let $\{\mathbf{A}_x : x \in X\}$ be an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. An antichain labeling (or ac-labeling) is a choice function $f \in \prod_{x \in X} A_x$ such that for all $x, y \in X$,

$$x < y \implies f(x) = 0 \text{ or } f(y) = 1.$$



$\mathbf{A}_x$



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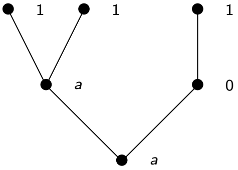

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Conuclei and conuclear images

If **A** is a residuated lattice, a map $\sigma\colon A \to A$ is a conucleus on **A** if for all $x, y \in A$:

- $\sigma(x) \leq x$
- $\sigma(\sigma(x)) = \sigma(x)$.
- $x \leq y$ implies $\sigma(x) \leq \sigma(y)$
- $\sigma(x)\sigma(y) \leq \sigma(xy)$
- $\sigma(x)\sigma(1) = \sigma(1)\sigma(x) = \sigma(x)$

If σ is a conucleus on $\mathbf{A} = (A, \wedge, \vee, \cdot, \to, 0, 1)$, then

$$\mathbf{A}_\sigma = (\sigma[A], \wedge_\sigma, \vee, \cdot, \to_\sigma, 0, \sigma(1))$$

is also a residuated lattice, where $x \wedge_\sigma y = \sigma(x \wedge y)$ and $x \to_\sigma y = \sigma(x \to y)$.

Poset products

Let $(X, \leq)$ be a poset and $\{\mathbf{A}_x : x \in X\}$ is an indexed collection of residuated lattices sharing a common least element 0 and common greatest element 1. Set $\mathbf{B} = \prod_{x \in X} \mathbf{A}_x$ and define a map $\Box \colon B \to B$ by

$$\Box(f)(x) = \begin{cases} f(x) & \text{if } f(y) = 1 \text{ for all } y > x \\ 0 & \text{if there exists } y > x \text{ with } f(y) \neq 1. \end{cases}$$

Then $\Box$ is a conucleus on the direct product. The conuclear image consists of antichain labelings and is the poset product of the indexed family:

$$\mathbf{B}_{\Box} = \prod_{(X,\leq)} \mathbf{A}_x.$$

Thinking about poset products

Poset products were originally introduced by P. Jipsen and. F. Montagna as a common generalization of direct products and nested sums (sometimes called ordinal sums).

- If $(X, =)$ is the index poset, then the poset product of $\{\mathbf{A}_x : x \in X\}$ is just the direct product.
- If $x < y$ in the poset $(\{x, y\}, \leq)$, then the poset product consists of the nested sum of $\mathbf{A}_x$ and $\mathbf{A}_y$ (intuitively obtained by replacing the unit of $\mathbf{A}_x$ by $\mathbf{A}_y$).

Poset products can be thought of as iterating the direct product and nested sum constructions.

Poset product representations

Recall that a GBL-algebra is a residuated lattice that satisfies divisibility ($x(x \to y) = x \wedge y$). Almost all of the past work on poset product representations has been directed at GBL-algebras and BL-algebras (the subvariety generated by totally ordered GBL-algebras).

Theorem (Jipsen-Montagna 2010):

- Every GBL-algebra embeds in a poset product of totally ordered MV-algebras.
- Every n-potent GBL-algebra (satisfying $x^{n+1} = x^n$) embeds into a poset product of finite simple n-potent MV-algebra chains.

Some definitions

Definition (idempotent center):

- The idempotent center of the residuated lattice **A** is the set $\mathcal{H}(A) = \{a \in A : a^2 = a\}$.
- If $\mathcal{H}(A)$ is a (necessarily Heyting) subalgebra of **A** and for all $i \in \mathcal{H}(A)$, $a \in A$ we have $ia = i \wedge a$, we say that it is a central subalgebra of **A** and denote it by $\mathcal{H}(\mathbf{A})$.

Definition (central filters):

- A filter of a residuated lattice **A** is a subset that is upward closed and closed under $\cdot$.
- For each subset S of A, there is a smallest filter containing S called the filter generated by S.
- A filter is called central if it is the filter generated the idempotent elements it contains.
- A value is completely meet irreducible element in the lattice of filters.

Centered residuated lattices

Representability by poset products of simple residuated lattices turns out to depend crucially on $\mathcal{H}(A)$ fitting inside **A** 'nicely':

Definition:

We say that a residuated lattice **A** is centered if:

- $\mathcal{H}(\mathbf{A})$ is a central subalgebra of **A**.
- Every filter of **A** is central.
- **A** satisfies the diamond condition: For every $i \in \mathcal{H}(A)$ and $a \in A$, there exists $j \in \mathcal{H}(A)$ such that $i \wedge j \leq a \leq i \vee j$.

Theorem (F.-Jipsen 2023+):

Every centered residuated lattice embeds into a poset product of simple residuated lattices, and is therefore isomorphic to an algebra of antichain labelings.

Multipotence

Recall that a residuated lattice is multipotent if for all a there exists $n \in \mathbb{N}$ such that $a^{n+1} = a^n$.

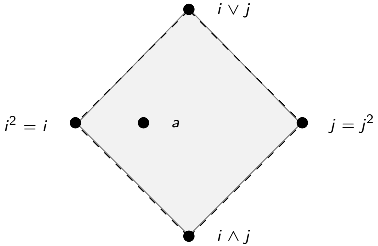
Lemma (F.-Jipsen 2023+):

The follow are equivalent for a residuated lattice $\mathbf{A}$.

1. Every filter of $\mathbf{A}$ is central.
2. $\mathbf{A}$ is multipotent.

The diamond condition

The diamond condition: For every $i \in \mathcal{H}(A)$ and $a \in A$, there exists $j \in \mathcal{H}(A)$ such that $i \wedge j \leq a \leq i \vee j$.



The Blok-Ferreirim theorems

The Blok-Ferreirim theorem has had an impact in the theory of hoops and GBL-algebras, and roughly states that subdirectly irreducibles can be decomposed as a nested/ordinal sum with a totally ordered algebra on top.

When all the filters are central, as in centered residuated lattices, we can give a particularly nice form of this theorem due to the diamond condition:

Blok-Ferreirim Theorem for Centered Residuated Lattices (F.-Jipsen 2023+):

Let $\mathbf{A}$ be a subdirectly irreducible centered residuated lattice. Then there is a maximum element m of $\mathcal{H}(A) \setminus \{1\}$, and for all $a \in A$ we have $m \leq a$ or $a \leq m$.

Sketch of the main proof

Let $\mathbf{A}$ be a centered residuated lattice. We will embed $\mathbf{A}$ in a poset product of simple residuated lattices.

Step 1: Let $(X, \subseteq)$ be the collection of values of $\mathbf{A}$ ordered by inclusion. Because all the filters of $\mathbf{A}$ are central, the lattice of filters of $\mathbf{A}$ is isomorphic to the lattice of filters of $\mathcal{H}(\mathbf{A})$ and we can just as well take the poset of values of $\mathcal{H}(\mathbf{A})$.

Step 2: For each $x \in X$, $\mathbf{A}/x$ is subdirectly irreducible because x is completely meet irreducible. The follow is not hard to show.

Lemma:

The class of centered residuated lattices is closed under quotients.

Hence, for each $x \in X$, $\mathbf{A}/x$ is a subdirectly irreducible centered residuated lattice.

Sketch of the main proof (cont)

Step 3: By the Blok-Ferreirim Theorem for centered residuated lattices, for each $\mathbf{A}/x$ there exists $m_x \in \mathcal{H}(\mathbf{A}/x)$ such that for all $a \in A/x$, $a \leq m_x$ or $m_x \leq a$. For each $x \in X$, define $A_x = \uparrow m_x$. Then A_x the universe of 0-free subalgebra of $\mathbf{A}/x$, so forms a residuated lattice $\mathbf{A}_x$.

Step 4: We claim that $\mathbf{A}$ embeds in $\prod_{(X,\subseteq)} \mathbf{A}_x$. The embedding is $a \mapsto [a](-)$, where for each $x \in X$,

$$[a](x) = \begin{cases} a/x & \text{if } m_x \leq a/x \\ 0 & \text{if } a/x < m_x. \end{cases}$$

The proof that $a \mapsto [a](-)$ is an embedding depends on the fact that $\mathcal{H}(\mathbf{A})$ is a central subalgebra of $\mathbf{A}$, together with $\mathbf{A}$ being multipotent (equivalent to each filter being central).

Varieties and logics

Centered residuated lattices don't form an especially nice class, and what we're interested in for logical purposes are varieties.

Definition:

For each $n \in \mathbb{N}$, let S_n denote the subvariety of residuated lattices axiomatized by:

- $a^n b = a^n \wedge b$.
- $a^n \to b^n = (a^n \to b^n)^2$.
- $a \leq b^n \vee (b^n \to a^n)$.

Further, for each $n \in \mathbb{N}$ denote by C_n the subvariety of S_n axiomatized by

$$(a \to b) \to (b \to a) = b \to a.$$

Concrete embedding theorems

Theorem (Jipsen-Montagna 2010):

For each $n \in \mathbb{N}$, the variety generated by poset products of simple n-potent MV-algebras chains is the variety of n-potent GBL-algebras.

Theorem (F.-Jipsen 2023+):

Let $n \in \mathbb{N}$.

- S_n is the variety generated by poset products of simple n-potent residuated lattices.
- C_n is the variety generated by poset products of simple n-potent totally ordered residuated lattice.

Thank you!

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