A General Method for Representing Sets of Relations by Vectors

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Introduction

The use of ROBDDs often leads to an amazing computational power of RelView, in particular, if the solution of a hard problem is based on the computation of a subset $\mathcal{R}$ of a powerset 2^X.

In certain situations X is a direct product $Y \times Z$, which means that $\mathcal{R}$ is a subset of $[Y \leftrightarrow Z]$, the set of relations between Y and Z.

Examples:

- Solutions of a timetabling problem (B. Kehden, R. B.).
- Computation of the set of up-closed multirelations (W. Guttmann).
- Computational proof of a variant of the Kuratowski closure-complement-theorem (R. B.).
- Experiments with sufficient criteria for the existence of kernels (R. B.).

We present a method for the relational computation of sets $\mathcal{R}$ of relations that generalizes the method presented at RAMiCS 2021.

As the method presented at RAMiCS 2021, the set $\mathcal{R}$ is computed as a vector $\mathfrak{r} : [X \leftrightarrow Y] \leftrightarrow \mathbf{1}$ that represents $\mathcal{R}$ as a subset of $[X \leftrightarrow Y]$.

In contrast with the old method, however, the new method

- does not specify R to belong to $\mathcal{R}$ by an inclusion $\mathfrak{v} \subseteq \mathfrak{w}$ between columnwise extendible vector expressions $\mathfrak{v}, \mathfrak{w} \in \mathbf{VE}(r)$ over a vector representation r of R
- but by an inclusion $\mathfrak{R} \subseteq \mathfrak{S}$ between general relation-algebraic expressions $\mathfrak{R}, \mathfrak{S} \in \mathbf{RE}(R)$ as usual.

Compared with the method presented at RAMiCS 2021 the proposed new method is **much simpler** and **straightforward**.

The Relational Constructions we Need

- **Basic operations** and **basic constants** of relation algebra.

$$R \cup S \qquad R \cap S \qquad R;S \qquad \overline{R}, \qquad R^{\mathsf{T}} \qquad \mathsf{O} \qquad \mathsf{L} \qquad \mathsf{I}$$

- **Projection relations** $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow Y$ of $X \times Y$ as further basic constants. Pointwise description:

$$\pi_{u,x} \iff u_1 = x \qquad \rho_{u,y} \iff u_2 = y$$

- **Left pairing** $[\![R,S]\!] := \pi;R \cap \rho;S$, **right pairing** $[R,S]\!] := [\![R^{\mathsf{T}},S^{\mathsf{T}}]^{\mathsf{T}}$ and **parallel composition** $R \,\|\, S := [\pi;R, \rho;S]\!]$ (derived operations). Pointwise descriptions:

$$[\![R,S]_{u,z} \iff R_{u_1,z} \wedge S_{u_2,z}$$
$$[R,S]\!]_{z,u} \iff R_{z,u_1} \wedge S_{z,u_2}$$
$$(R \,\|\, S)_{u,v} \iff R_{u_1,v_1} \wedge S_{u_2,v_2}$$

- **Membership relations** $\mathsf{M} : X \leftrightarrow 2^X$ as further basic constants. Pointwise description:

$$\mathsf{M}_{x,Y} \iff x \in Y$$

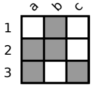
If X is a direct product $Y \times Z$, we use $\mathbf{M}$ instead of M and get:

$$\mathbf{M}_{u,R} \iff R_{u_1,u_2}$$

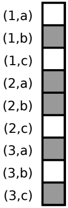
- **Vector representation** $\mathrm{vec}(R) = [\![R, \mathsf{I}]\!] ; \mathsf{L}$ (derived operation). Pointwise description:

$$\mathrm{vec}(R)_u \iff R_{u_1,u_2}$$

Example (produced by RelView):



$R : X \leftrightarrow Y$



$\mathrm{vec}(R) : X \times Y \leftrightarrow \mathbf{1}\ (= \{\bot\})$

Columnwise Extendible Vector Expressions

For a variable r of type $X \leftrightarrow \mathbf{1}$, the set $\mathbf{VE}(r)$ of typed **columnwise extendible vector expression** over r is inductively defined as follows:

- We have $r \in \mathbf{VE}(r)$ and its type is $X \leftrightarrow \mathbf{1}$.
- If $v : Y \leftrightarrow \mathbf{1}$ is a vector, then $v \in \mathbf{VE}(r)$ and its type is $Y \leftrightarrow \mathbf{1}$.
- If $\mathfrak{w} \in \mathbf{VE}(r)$ has type $Y \leftrightarrow \mathbf{1}$, then $\overline{\mathfrak{w}} \in \mathbf{VE}(r)$ and its type is $Y \leftrightarrow \mathbf{1}$.
- If $\mathfrak{w}, \mathfrak{u} \in \mathbf{VE}(r)$ have both type $Y \leftrightarrow \mathbf{1}$, then $\mathfrak{w} \cup \mathfrak{u} \in \mathbf{VE}(r)$ and $\mathfrak{w} \cap \mathfrak{u} \in \mathbf{VE}(r)$ and their types are $Y \leftrightarrow \mathbf{1}$.
- If $\mathfrak{w} \in \mathbf{VE}(r)$ has type $Y \leftrightarrow \mathbf{1}$ and $\mathfrak{R}$ is a relation-algebraic expression of type $Z \leftrightarrow Y$ **free of r**, then $\mathfrak{R}; \mathfrak{w} \in \mathbf{VE}(r)$ and its type is $Z \leftrightarrow \mathbf{1}$.

Examples:

$$r \qquad R^{\mathsf{T}}; (S; r \cup v) \qquad [\![r, R; r]\!] \;\; (= \pi; r \cap \rho; R; r)$$

For all variables r of type $X \leftrightarrow \mathbf{1}$, relations $R : X \leftrightarrow Z$ and $\mathfrak{w} \in \mathbf{VE}(r)$ we denote by $\mathfrak{w}[R/r]$ the **replacement** of r by R in $\mathfrak{w}$.

Inductive definition:

- $r[R/r] = R$.
- $v[R/r] = v; \mathsf{L}$, with $\mathsf{L} : \mathbf{1} \leftrightarrow Z$.
- $\overline{\mathfrak{w}}[R/r] = \overline{\mathfrak{w}[R/r]}$,
- $(\mathfrak{w} \cup \mathfrak{u})[R/r] = \mathfrak{w}[R/r] \cup \mathfrak{u}[R/r]$ and $(\mathfrak{w} \cap \mathfrak{u})[R/r] = \mathfrak{w}[R/r] \cap \mathfrak{u}[R/r]$.
- $(\mathfrak{R}; \mathfrak{w})[R/r] = \mathfrak{R}; (\mathfrak{w}[R/r])$.

Examples: For r of type $X \leftrightarrow \mathbf{1}$ and $\mathsf{M} : X \leftrightarrow 2^X$ we get:

$$\begin{aligned} r[\mathsf{M}/r] &= \mathsf{M} \\ R^{\mathsf{T}}; (S; r \cup v)[\mathsf{M}/r] &= R^{\mathsf{T}}; (S; \mathsf{M} \cup v; \mathsf{L}) \\ [\![r, R; r]\!][\mathsf{M}/r] &= [\![\mathsf{M}, R; \mathsf{M}]\!] \end{aligned}$$

If $\mathfrak{w}$ has type $Y \leftrightarrow \mathbf{1}$, then $\mathfrak{w}[R/r]$ has type $Y \leftrightarrow Z$.

The Method of RAMiCS 2021

Recall the function

$$\mathrm{vec} : [X \leftrightarrow Y] \to [X \times Y \leftrightarrow \mathbf{1}] \qquad \mathrm{vec}(R) = [\![R, \mathsf{I}]\!] ; \mathsf{L}.$$

Together with the function (with π and ρ as projection relations of $X \times Y$)

$$\mathrm{Rel} : [X \times Y \leftrightarrow \mathbf{1}] \to [X \leftrightarrow Y] \qquad \mathrm{Rel}(r) = \pi^{\mathsf{T}} ; (r ; \mathsf{L} \cap \rho)$$

it forms a Boolean lattice isomorphism between $[X \leftrightarrow Y]$ and $[X \times Y \leftrightarrow \mathbf{1}]$.

Theorem 1. Assume the subset $\mathcal{R}$ of $[X \leftrightarrow Y]$ to be specified as

$$\mathcal{R} = \{\mathrm{Rel}(r) \mid r \in [X \times Y \leftrightarrow \mathbf{1}] \wedge \mathfrak{v} \subseteq \mathfrak{w}\},$$

where $\mathfrak{v}, \mathfrak{w} \in \mathbf{VE}(r)$. By means of $\mathbf{M} : X \times Y \leftrightarrow [X \leftrightarrow Y]$ we get a vector $\mathfrak{r} : [X \leftrightarrow Y] \leftrightarrow \mathbf{1}$ that represents $\mathcal{R}$ as

$$\mathfrak{r} := \overline{\mathsf{L} ; (\mathfrak{v}[\mathbf{M}/r] \cap \overline{\mathfrak{w}[\mathbf{M}/r]})}^{\mathsf{T}}.$$

The method proved to be superior to a development of the vector representation $\mathfrak{r} : [X \leftrightarrow Y] \leftrightarrow \mathbf{1}$ from a logical specification of R to belong to $\mathcal{R}$.

Drawback: A specification of R to belong to $\mathcal{R}$ by means of its vector representation $r : X \times Y \leftrightarrow \mathbf{1}$ quite often is "unnatural".

Some "natural" specifications:

- Set of **antisymmetric** relations:

$$\mathcal{R} = \{R \mid R \in [X \leftrightarrow X] \wedge R \cap R^{\mathsf{T}} \subseteq \mathsf{I}\}$$

- Set of **transitive** relations:

$$\mathcal{R} = \{R \mid R \in [X \leftrightarrow X] \wedge R; R \subseteq R\}$$

- Set of **rectangular** relations:

$$\mathcal{R} = \{R \mid R \in [X \leftrightarrow Y] \wedge R; \mathsf{L}; R \subseteq R\}$$

- Set of **Ferrers** relations:

$$\mathcal{R} = \{R \mid R \in [X \leftrightarrow Y] \wedge R; \overline{R}^{\mathsf{T}}; R \subseteq R\}$$

We want to start with such specifications.

Properties of Vector Representations

That the function vec is a Boolean lattice isomorphism means:

$$\mathrm{vec}(\overline{R}) = \overline{\mathrm{vec}(R)}$$
$$\mathrm{vec}(R \cup S) = \mathrm{vec}(R) \cup \mathrm{vec}(S)$$
$$\mathrm{vec}(R \cap S) = \mathrm{vec}(R) \cap \mathrm{vec}(S)$$

Concerning transposition, it is known that

$$\mathrm{vec}(R^{\mathsf{T}}) = \mathsf{S}; \mathrm{vec}(R),$$

where $\mathsf{S} := [\rho, \pi]\!] : X \times Y \leftrightarrow Y \times X$ exchanges the components of a pair

Concerning composition, B. Kehden proved (RelMiCS/AKA 2006)

$$\mathrm{vec}(Q; R; S) = (Q \parallel S^{\mathsf{T}}); \mathrm{vec}(R).$$

For a generalization of B. Kehden's result we consider:

$$P := \mathsf{I} \cap \pi;\rho;\pi^{\mathsf{T}};\rho^{\mathsf{T}} : (X\times Y)\times(Y\times Z) \leftrightarrow (X\times Y)\times(Y\times Z)$$

The partial mapping (univalent relation) P acts as a filter when composing it with a suitable relation as, using partial function notation,

$$P((x,y),(y',z)) = \begin{cases} ((x,y),(y,z)) & \text{if } y = y' \\ \text{undefined} & \text{if } y \neq y'. \end{cases}$$

For the partial mapping $\mathsf{C} := P;(\pi \,\|\, \rho) : (X\times Y)\times(Y\times Z) \leftrightarrow X\times Z$ we get

$$\mathsf{C}((x,y),(y',z)) = \begin{cases} (x,z) & \text{if } y = y' \\ \text{undefined} & \text{if } y \neq y', \end{cases}$$

again using partial function notation. This suggests

$$\mathrm{vec}(Q;R) = \mathsf{C}^{\mathsf{T}};[\![\mathrm{vec}(Q),\mathrm{vec}(R)]\!]$$

and in fact we have been able to prove this equation relation-algebraically.

The New Method

The last equations allow to compute vector representations via a recursive function

$$\nu_r : \mathbf{RE}(R) \to \mathbf{VE}(r),$$

where r and R are variables of type $X \times Y \leftrightarrow \mathbf{1}$ and $X \leftrightarrow Y$, respectively.

- $\nu_r(R) = r$.
- $\nu_r(S) = \mathrm{vec}(S)$ for all relations S.
- $\nu_r(\overline{\mathfrak{R}}) = \overline{\nu_r(\mathfrak{R})}$.
- $\nu_r(\mathfrak{R}^{\mathsf{T}}) = \mathsf{S}; \nu_r(\mathfrak{R})$.
- $\nu_r(\mathfrak{R} \cup \mathfrak{S}) = \nu_r(\mathfrak{R}) \cup \nu_r(\mathfrak{S})$.
- $\nu_r(\mathfrak{R} \cap \mathfrak{S}) = \nu_r(\mathfrak{R}) \cap \nu_r(\mathfrak{S})$.
- $\nu_r(\mathfrak{R}; \mathfrak{S}) = \mathsf{C}^{\mathsf{T}}; [\![\nu_r(\mathfrak{R}), \nu_r(\mathfrak{S})]\!]$.

Recall that $\mathbf{RE}(R)$ is the set of relation-algebraic expressions built from R and $\mathbf{VE}(r)$ is the set of columnwise extendible vector expressions over r.

Lemma. Let be r and R variables of type $X \times Y \leftrightarrow \mathbf{1}$ and $X \leftrightarrow Y$, respectively. For all $\mathfrak{R} \in \mathbf{RE}(R)$ we then have:

- $\nu_r(\mathfrak{R}) \in \mathbf{VE}(r)$.
- If r is instantiated as $\mathrm{vec}(R)$, then $\nu_r(\mathfrak{R}) = \mathrm{vec}(\mathfrak{R})$.

Proof by induction on the structure of $\mathfrak{R}$.

Theorem 2. Assume the subset $\mathcal{R}$ of $[X \leftrightarrow Y]$ to be specified as

$$\mathcal{R} = \{R \mid R \in [X \leftrightarrow Y] \wedge \mathfrak{R} \subseteq \mathfrak{S}\},$$

where $\mathfrak{R}, \mathfrak{S} \in \mathbf{RE}(R)$, and let r be any variable of type $X \times Y \leftrightarrow \mathbf{1}$. By means of $\mathbf{M} : X \times Y \leftrightarrow [X \leftrightarrow Y]$ we then get a vector $\mathfrak{r} : [X \leftrightarrow Y] \leftrightarrow \mathbf{1}$ that represents $\mathcal{R}$ as

$$\mathfrak{r} := \overline{\mathsf{L}; (\nu_r(\mathfrak{R})[\mathbf{M}/r] \cap \overline{\nu_r(\mathfrak{S})[\mathbf{M}/r]})}^{\mathsf{T}}.$$

Proof. First, we prove that the original specification of $\mathcal{R}$ is equivalent to the specification

$$\begin{aligned}\mathcal{R} &= \{R \mid \exists\, r : r \in [X \times Y \leftrightarrow \mathbf{1}] \wedge R = \mathrm{Rel}(r) \wedge \nu_r(\mathfrak{R}) \subseteq \nu_r(\mathfrak{S})\} \\ &= \{\mathrm{Rel}(r) \mid r \in [X \times Y \leftrightarrow \mathbf{1}] \wedge \nu_r(\mathfrak{R}) \subseteq \nu_r(\mathfrak{S})\}.\end{aligned}$$

Let an arbitrary $R : X \leftrightarrow Y$ be given. Then we have

$$\begin{aligned}\mathfrak{R} \subseteq \mathfrak{S} &\iff \mathrm{vec}(\mathfrak{R}) \subseteq \mathrm{vec}(\mathfrak{S}) \\ &\iff \nu_{\mathrm{vec}(R)}(\mathfrak{R}) \subseteq \nu_{\mathrm{vec}(R)}(\mathfrak{S}) \\ &\iff \exists\, r : r \in [X \times Y \leftrightarrow \mathbf{1}] \wedge r = \mathrm{vec}(R) \wedge \nu_r(\mathfrak{R}) \subseteq \nu_r(\mathfrak{S}) \\ &\iff \exists\, r : r \in [X \times Y \leftrightarrow \mathbf{1}] \wedge R = \mathrm{Rel}(r) \wedge \nu_r(\mathfrak{R}) \subseteq \nu_r(\mathfrak{S}),\end{aligned}$$

where we use the lemma and that vec is a Boolean lattice isomprphism.

From the lemma we also get $\nu_r(\mathfrak{R}) \in \mathbf{VE}(r)$ and $\nu_r(\mathfrak{S}) \in \mathbf{VE}(r)$.

Hence, Theorem 1 is applicable and yields the desired result.

Example: Antisymmetric Relations

We consider antisymmetric relations, i.e., we use the specification

$$\mathcal{R} = \{R \mid R \in [X \leftrightarrow X] \wedge R \cap R^{\mathsf{T}} \subseteq \mathsf{I}\}.$$

Given any variable r of type $X \times X \leftrightarrow \mathbf{1}$ we get

- for the left-hand side of $R \cap R^{\mathsf{T}} \subseteq \mathsf{I}$ that

$$\nu_r(R \cap R^{\mathsf{T}}) = \nu_r(R) \cap \nu_r(R^{\mathsf{T}}) = \nu_r(R) \cap \mathsf{S}; \nu_r(R) = r \cap \mathsf{S}; r$$

  such that $\nu_r(R \cap R^{\mathsf{T}})[\mathbf{M}/r] = \mathbf{M} \cap \mathsf{S}; \mathbf{M}$,
- for the right-hand side of $R \cap R^{\mathsf{T}} \subseteq \mathsf{I}$ that

$$\nu_r(\mathsf{I}) = \mathrm{vec}(\mathsf{I})$$

  such that $\nu_r(\mathsf{I})[\mathbf{M}/r] = \mathrm{vec}(\mathsf{I}); \mathsf{L}$, where $\mathsf{L} : \mathbf{1} \leftrightarrow [X \leftrightarrow X]$.

Vector representation of the set of antisymmetric relations on X:

$$\mathfrak{antisymm} := \overline{\mathsf{L}; (\mathbf{M} \cap \mathsf{S}; \mathbf{M} \cap \overline{\mathrm{vec}(\mathsf{I}); \mathsf{L}})}^{\mathsf{T}}$$

Example: Transitive Relations

We consider transitive relations, i.e., we use the specification

$$\mathcal{R} = \{R \mid R \in [X \leftrightarrow X] \wedge R;R \subseteq R\}.$$

Given any variable r of type $X \times X \leftrightarrow \mathbf{1}$ we get

- for the left-hand side of $R;R \subseteq R$ that

$$\nu_r(R;R) = \mathsf{C}^\mathsf{T};[\![\nu_r(R), \nu_r(R)]\!] = \mathsf{C}^\mathsf{T};[\![r, r]\!],$$

  such that $\nu_r(R;R)[\mathbf{M}/r] = \mathsf{C}^\mathsf{T};[\![\mathbf{M}, \mathbf{M}]\!]$,
- for the right-hand side of $R;R \subseteq R$ that

$$\nu_r(R) = r$$

  such that $\nu_r(R)[\mathbf{M}/r] = \mathbf{M}$.

Vector representation of the set of transitive relations on X:

$$\mathfrak{trans} := \overline{\mathsf{L};(\mathsf{C}^\mathsf{T};[\![\mathbf{M}, \mathbf{M}]\!] \cap \overline{\mathbf{M}})}^\mathsf{T}$$

Implementation

RelView-programs for the two projection relations $\pi : X \times Y \leftrightarrow X$ and $\rho : X \times Y \leftrightarrow Y$, where the parameter $A : X \leftrightarrow Y$ provides the sets X and Y:

```
pr1(A)
  DECL XY = PROD(A*A^,A^*A)
  BEG  RETURN p-1(XY) END.                { XY <-> X }

pr2(A)
  DECL XY = PROD(A*A^,A^*A)
  BEG  RETURN p-2(XY) END.                { XY <-> Y }
```

RelView-function for the vector representation $\mathrm{vec}(R) : X \times Y \leftrightarrow \mathbf{1}$ of $R : X \leftrightarrow Y$:

```
vec(R) = dom([|R,I(R^*R)]).    { XY <-> 1 }
```

RelView-program for the parallel composition $R \| S$ of $R : X \leftrightarrow Y$ and $S : X' \leftrightarrow Y'$:

```
par(R,S)
  DECL A
  BEG  A = Ln1(R)*Ln1(S)^               { X <-> X' }
       RETURN [pr1(A)*R,pr2(A)*S|]      { XX' <-> YY' }
 END.
```

RelView-function for the relation $\mathsf{S} : X \times Y \leftrightarrow Y \times X$, where the parameter $A : X \leftrightarrow Y$ provides the sets X and Y:

```
Swap(A) = [pr2(A),pr1(A)|].             { XY <-> YX }
```

RelView-program for the relation $\mathsf{C} : (X \times Y) \times (Y \times Z) \leftrightarrow X \times Z$, where the parameters $A : X \leftrightarrow Y$ and $B : Y \leftrightarrow Z$ provide the sets X, Y, and Z:

```
Comp(A,B)
  DECL p1, p2, q1, q2, r1, r2, H
  BEG  q1 = pr1(A);                  { XY <-> X }
       q2 = pr2(A);                  { XY <-> Y }
       r1 = pr1(B);                  { YZ <-> Y }
       r2 = pr2(B);                  { YZ <-> Z }
       p1 = pr1(q2*r1^);             { XY YZ <-> XY }
       p2 = pr2(q2*r1^);             { XY YZ <-> YZ }
       H = p1*q2*r1^*p2^             { XY YZ <-> XY YZ }
       RETURN (I(H) & H)*par(q1,r2) { XY YZ <-> XZ }
  END.
```

RelView-programs for the sets of antisymmetric and transitive relations on X, where X is provided by the parameter $A : X \leftrightarrow X$:

```
Antisymm(A)
  DECL M, R, i
  BEG  M = epsi(pr1(A));              { XX <-> [X<->X] }
       R = M & Swap(A)*M;             { XX <-> [X<->X] }
       i = vec(I(A))*L1n(M)           { XX <-> [X<->X] }
       RETURN -(Ln1(R)^*(R & -i))^    { [X<->X] <-> 1 }
  END.

Trans(A)
  DECL M, R
  BEG  M = epsi(pr1(A));              { XX <-> [X<->X] }
       R = Comp(A,A)^*[|M,M]          { XX <-> [X<->X] }
       RETURN -(Ln1(R)^*(R & -M))^    { [X<->X] <-> 1 }
  END.
```

Conclusion

We have applied our method to many other classes of relations, e.g.,

- relations having kernels
- criteria for the existence of kernels
- many of the examples presented in the RAMiCS 2021 paper
- classes of relations not treated so far, e.g.,
  - lattices,
  - bounded partial orders,
  - finite directed acyclic graphs and arborescences,
  - tournaments,
  - difunctional relations,
  - strongly connected relations
  - maps having fixpoints.

What are those properties $P(R)$ of relations $R : X \leftrightarrow Y$ which can be expressed by a finite set of inclusions $\mathfrak{R}_i \subseteq \mathfrak{S}_i$ with $\mathfrak{R}_i, \mathfrak{S}_i \in \mathbf{RE}(R)$ for all i?