Amalgamation property for some varieties of BL-algebras generated by one finite set of BL-chains with finitely many components

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(joint work with Stefano Aguzzoli)

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Definition 1

A BL-algebra is a commutative, integral, bounded, prelinar, divisible residuated lattice of the form $\mathcal{A} = (A, *, \rightarrow, \wedge, \vee, 0, 1)$. A totally ordered BL-algebra is called BL-chain.

The class of all BL-algebras forms an algebraic variety, called $\mathbb{BL}$. Given a variety $\mathbb{L}$ of BL-algebras, with $Ch(\mathbb{L})$ we denote the class of the chains in $\mathbb{L}$.

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Theorem 2 ([AM03])

Every BL-chain can be uniquely decomposed (up to isomorphisms) as an ordinal sum of totally ordered Wajsberg hoops, with the first bounded.

BL-algebras and BL

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Let $\mathcal{A} = \bigoplus_{i\in I} \mathcal{A}_i$ be a BL-chain such that I is finite. We define $\#\mathcal{A} \stackrel{\text{def}}{=} |I|$.

Amalgamation property and one-sided amalgamation property

Definition 3

Let $\mathbb{L}$ be a class of algebras. A V-formation is a tuple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, i, j)$ such that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{L}$, $\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{B}$, and $\mathcal{A} \overset{j}{\hookrightarrow} \mathcal{C}$.

- We say that $\mathbb{L}$ has the *one-sided amalgamation property* (1AP), whenever for every V-formation $(\mathcal{A}, \mathcal{B}, \mathcal{C}, i, j)$ there is a tuple $(\mathcal{D}, h, k)$, called 1-amalgam, such that $\mathcal{D} \in \mathbb{L}$, $\mathcal{B} \overset{h}{\hookrightarrow} \mathcal{D}$, k is a homomorphism from $\mathcal{C}$ to $\mathcal{D}$, and $h \circ i = k \circ j$.
- We say that $\mathbb{L}$ has the *amalgamation property* (AP), whenever for every V-formation $(\mathcal{A}, \mathcal{B}, \mathcal{C}, i, j)$ there is a tuple $(\mathcal{D}, h, k)$, called an amalgam, such that $\mathcal{D} \in \mathbb{L}$, $\mathcal{B} \overset{h}{\hookrightarrow} \mathcal{D}$, $\mathcal{C} \overset{k}{\hookrightarrow} \mathcal{D}$, and $h \circ i = k \circ j$.

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For the varieties of BL-algebras a sufficient condition for the AP is the following.

Theorem 4 ([Mon06], [MMT14])

Let $\mathbb{L}$ be a non-trivial variety of BL-algebras. If $Ch(\mathbb{L})$ enjoys the AP then the same holds for $\mathbb{L}$.

Amalgamation property and one-sided amalgamation property: some results

Clearly the AP implies the 1AP. Interestingly, also the converse holds, if we assume that the class of algebras $\mathbb{L}$ satisfies some properties. By $\mathbb{L}_{\mathsf{FSI}}$ we denote the class of finitely subdirectly irreducible algebras of $\mathbb{L}$.

Theorem 5 ([FM22])

Let $\mathbb{L}$ be a variety with the congruence extension property such that $\mathbb{L}_{FSI}$ is closed under subalgebras. The following are equivalent:

- *$\mathbb{L}$ has the amalgamation property.*
- *$\mathbb{L}$ has the one-sided amalgamation property.*
- *$\mathbb{L}_{FSI}$ has the one-sided amalgamation property.*
- *Every V-formation of finitely generated algebras from $\mathbb{L}_{FSI}$ has an amalgam in $\mathbb{L}_{FSI} \times \mathbb{L}_{FSI} = \{\mathcal{A} \times \mathcal{B} : \mathcal{A}, \mathcal{B} \in \mathbb{L}_{FSI}\}$.*
- *Every V-formation of finitely generated algebras from $\mathbb{L}_{FSI}$ has an amalgam in $\mathbb{L}$.*

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- *Every V-formation of finitely generated algebras from $\mathbb{L}_{FSI}$ has an amalgam in $\mathbb{L}$.*

Theorem 6 ([AB23])

A variety $\mathbb{L}$ of BL-algebras has the AP if and only if $Ch(\mathbb{L})$ has the 1AP.

The AP for varieties of MV-algebras has already been classified.

Theorem 7 ([NL00])

A variety $\mathbb{L}$ *of MV-algebras has the AP if and only if it is single-chain generated, i.e.* $\mathbb{L} = \mathbf{V}(\mathcal{A})$*, for some* $\mathcal{A} \in Ch(\mathcal{A})$.

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Moving to BL-algebras.

Theorem 8 ([Mon06])

$Ch(\mathbb{BL})$ *has the AP, and hence the variety of BL-algebras has the AP.*

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Differently from MV-algebras, it is not true that a variety of BL-algebras has the AP iff it is single-chain generated.

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For every $n \geq 4$ the variety $\mathbb{G}_n$ generated by the n-element Gödel-chain, does not have the AP.

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For every $n \geq 4$ the variety $\mathbb{G}_n$ generated by the n-element Gödel-chain, does not have the AP.

In [AB21] we classified the AP for varieties of BL-algebras which are generated by one chain with finitely-many components.

AP for some varieties of BL-algebras and open problems

Theorem 10 ([AB21])

Let $\mathbb{L}$ be a variety of BL-algebras generated by one chain with finitely many components. Then the following are equivalent:

(i) $\mathbb{L}$ *has the AP.*

(ii) *Every BL-chain* $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$ *such that* $\mathbf{V}(\mathcal{A}) = \mathbb{L}$ *satisfies the following conditions.*

- $|I| \leq 3$.
- *There is at most one* $i \in I \setminus \{0\}$ *such that* $\mathcal{A}_i$ *is infinite, and there is at most one* $j \in I \setminus \{0\}$ *such that* $\mathcal{A}_j$ *is bounded.*
- *If* $|I| \geq 2$ *then the following ones hold.*
  - *If* $\mathcal{A}_0$ *has infinite rank, then* $\mathcal{L}_k \hookrightarrow \mathcal{A}_0$, *for every* $k \geq 2$.
  - *If* $\mathcal{A}_0$ *is infinite and* $\text{rank}(\mathcal{A}_0) = k$, *then* $\mathcal{L}_k \hookrightarrow \mathcal{A}_0$.

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In this talk we provide a partial answer, by classifying the AP for varieties generated by one fine set of BL-chains with finitely-many components that are either finite Wajsberg hoops or cancellative hoops.

Main results

Theorem 12 ([AB23])

Let $\mathbb{L}$ be a variety generated by one finite set of BL-chains with finitely many components, that are either finite or cancellative. Then $\mathbb{L}$ has the AP if and only if one of the following two cases holds.

- $\mathbb{L} = \mathbf{V}(\mathcal{A})$, *where* $\mathcal{A} \in Ch(\mathbb{L})$, *and satisfies one of the following conditions:*
  - a) $\#\mathcal{A} \leq 2$, *there is at most one cancellative component, and the others are finite (including the first-one).*
  - b) $\#\mathcal{A} = 3$, *two components (including the first-one) are finite, and the other one is cancellative.*
- $\mathbb{L} = \mathbf{V}(\{\mathcal{B}, \mathcal{C}\})$, *where:*
  - c) $\mathcal{B}, \mathcal{C} \in Ch(\mathbb{L})$, $\#\mathcal{B} = \#\mathcal{C} = 2$.
  - d) $(\mathcal{B})_1$ *is finite,* $(\mathcal{C})_1$ *is cancellative, and* $(\mathcal{B})_0 \simeq (\mathcal{C})_0$.

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Corollary 13 ([AB23])

Let S be a finite set of finite BL-chains. Then $\mathbf{V}(S)$ *has the AP if and only there exist a finite BL-chain $\mathcal{A}$ such that* $\#\mathcal{A} \leq 2$, *and* $S = \{\mathcal{A}\}$.

Main theorem: proof sketch

It can be shown that there exists an ▸ m-set S such that $\mathbf{V}(S) = \mathbb{L}$. We have two cases.

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Lemma 14

Let S be an m-set in which every chain has either cancellative or finite components. Suppose that at least one of the following conditions holds:

- *There are $\mathcal{A}, \mathcal{B} \in S$ such that $\#\mathcal{A} = \#\mathcal{B} = k$, and $\mathcal{A} \neq \mathcal{B}$, with $k \geq 1$, $k \neq 2$.*
- *There are $\mathcal{A}, \mathcal{B} \in S$ such that $\#\mathcal{A} = \#\mathcal{B} = 2$, and $\mathcal{A} \neq \mathcal{B}$, where either $\mathcal{A}, \mathcal{B}$ are both finite or $(\mathcal{A})_1, (\mathcal{B})_1$ are both cancellative.*
- *There are $\mathcal{A}, \mathcal{B} \in S$ such that $\#\mathcal{A} = 2$ and $\#\mathcal{B} = 3$.*

Then $\mathbf{V}(S)$ does not have the AP.

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Then $\mathbf{V}(S)$ does not have the AP.

There are a number of cases to check, but using these results (and others) it can be shown that $Ch(\mathbb{L})$ has the 1AP if and only if $S = \{\mathcal{B}, \mathcal{C}\}$, where:

c) $\mathcal{B}, \mathcal{C} \in Ch(\mathbb{L})$, $\#\mathcal{B} = \#\mathcal{C} = 2$.

d) $(\mathcal{B})_1$ is finite, $(\mathcal{C})_1$ is cancellative, and $(\mathcal{B})_0 \simeq (\mathcal{C})_0$.

The proof is settled.

Open problems

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One of the main issues with this general case concerns the following problem:

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Let $\mathcal{A}, \mathcal{B}$ be two non-trivial MV-chains. When is it possible to define a homomorphism from $\mathcal{A}$ to $\mathcal{B}$?

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Nevertheless, we have a partial result.

Lemma 17 ([AB23])

Let S be an m-set containing a BL-chain $\mathcal{A}$ such that $\#\mathcal{A} \geq 2$, and one of the following holds:

- $(\mathcal{A})_0$ *has infinite rank and* $\mathcal{L}_n \not\to (\mathcal{A})_0$*, for some* $n \in \mathbb{N}$.
- $(\mathcal{A})_0$ *is infinite, has rank n, and* $\mathcal{L}_n \not\to (\mathcal{A})_0$.

Then $\mathbf{V}(S)$ *does not have the AP.*

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APPENDIX

Axiomatization of BL

The basic connective are $\{\&, \to, \bot\}$ (formulas built inductively: a theory is a set of formulas). Useful derived connectives are the following ones:

(negation) $\neg\varphi \stackrel{\text{def}}{=} \varphi \to \bot$

(conjunction) $\varphi \wedge \psi \stackrel{\text{def}}{=} \varphi\&(\varphi \to \psi)$

(disjunction) $\varphi \vee \psi \stackrel{\text{def}}{=} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$

(top) $\top \stackrel{\text{def}}{=} \neg\bot$

MTL can be axiomatized by using these axioms and modus ponens: $\frac{\varphi \quad \varphi\to\psi}{\psi}$.

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $(\varphi\&\psi) \to \varphi$

(A3) $(\varphi\&\psi) \to (\psi\&\varphi)$

(A4) $(\varphi\&(\varphi \to \psi)) \to (\psi\&(\psi \to \varphi))$

(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi)$

(A5b) $((\varphi\&\psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

(A7) $\bot \to \varphi$

back

Hoops

Definition 18 ([BF00, AFM07])

A *hoop* is a structure $\mathcal{A} = \langle A, *, \to, 1\rangle$ such that $\langle A, *, 1\rangle$ is a commutative monoid, and $\to$ is a binary operation such that

$$x \to x = 1, \quad x \to (y \to z) = (x * y) \to z \quad \text{and} \quad x * (x \to y) = y * (y \to x).$$

Definition 19

A *bounded* hoop is a hoop whose language is expanded with a constant 0 such that $0 \leq x$, for every element x; conversely, an *unbounded* hoop is a hoop without minimum.

Proposition 1 ([BF00, AFM07])

- *A hoop is Wajsberg iff it satisfies the equation* $(x \to y) \to y = (y \to x) \to x$.
- *A hoop is cancellative iff it satisfies the equation* $x = y \to (x * y)$.
- *Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with MV-algebras.*

back

- Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a totally ordered Wajsberg hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded.
- Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of $\mathcal{A}_0$, whose top is 1, and whose operations are

$$x \to y = \begin{cases} x \to^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

$$x * y = \begin{cases} x *^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\ y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\}, x \in A_j) \end{cases}$$

- As a consequence, if $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$ then $x < y$.

back

BL-algebras

A BL-algebra is an algebra $\langle A, *, \to, \wedge, \vee, 0, 1\rangle$. such that:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1\rangle$ is a commutative monoid.
3. $\langle *, \to\rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \to y$ for all $x, y, z \in A$. In particular, it holds that $x \to y = \max\{z \in A : z * x \leq y\}$.
4. The following equations hold.

(Prelinearity) $\qquad (x \to y) \vee (y \to x) = 1.$

(Divisibility) $\qquad (x \wedge y) = x * (x \to y).$

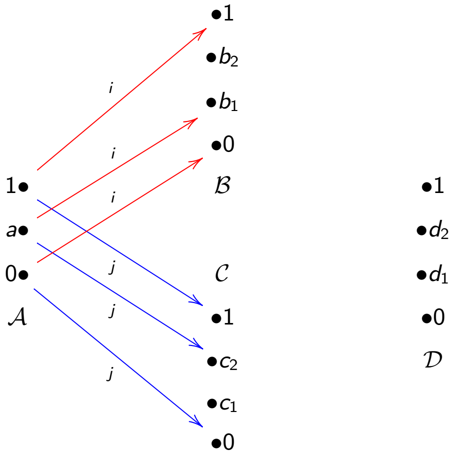
A totally ordered BL-algebra is called *BL-chain*.

- The class of BL-algebras forms a variety, called $\mathbb{BL}$. The logic corresponding to BL-algebras is called ▸ BL.
- An axiomatic extension of BL is a logic obtained by adding other axioms to it.
- Every axiomatic extension of BL is algebraizable in the sense of [BP89], and hence every subvariety of $\mathbb{BL}$ induces a logic.

◂ back

The failure of the AP for $Ch(\mathbb{G}_4)$

Pick the V-formation $(\mathcal{A}, \mathcal{B}, \mathcal{C}, i, j)$, with $\mathcal{A}, \mathcal{B}, \mathcal{C} \in Ch(\mathbb{G}_4)$, defined as in the picture.



Since every chain in $\mathbb{G}_4$ has at most 4 elements, then there is no 1-amalgam in $Ch(\mathbb{G}_4)$.

back

Minimal set of generators and a partial list of their properties

Definition 20

A *minimal set of generators* (m-set) is a non-empty finite set of non-trivial BL-chains with finitely many components, say S, such that $\mathbf{V}(T) \subsetneq \mathbf{V}(S)$, for every $T \subsetneq S$.

Lemma 21

Let S be a non-empty finite set of non-trivial BL-chains with finitely many components. Then there exists an m-set (not necessarily unique) $T \subseteq S$ such that $\mathbf{V}(T) = \mathbf{V}(S)$.

Lemma 22

Let $\mathbb{L}$ be a variety generated by one finite set of BL-chains with finitely many components. Then,

- *There exists an m-set S such that $\mathbf{V}(S) = \mathbb{L}$.*
- *If $\mathbb{L}$ is single-chain generated, then $|S| = 1$, for every m-set S such that $\mathbf{V}(S) = \mathbb{L}$.*
- *If every chain in $\mathbb{L}$ is finite, then there is an m-set S containing only finite BL-chains such that $\mathbf{V}(S) = \mathbb{L}$.*
- *Let S be an m-set such that $\mathbf{V}(S) = \mathbb{L}$. For every $\mathcal{A} \in S$, if $\mathcal{A} \hookrightarrow \mathcal{B}$, with $\mathcal{B} \in Ch(\mathbb{L})$, then $\mathcal{B} \in Ch(\mathcal{A})$.*

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Proof of the main theorem: some technical lemmas - 1

Proposition 2

Let $\mathbb{L}$ be a variety of BL-algebras such that every chain has finitely many components. If $\mathbb{L}$ has the AP, then every $\mathcal{A} \in Ch(\mathbb{L})$ satisfies the following properties.

- $\#\mathcal{A} \leq 3$.
- *If $\mathcal{A}$ is finite, then $\#\mathcal{A} \leq 2$.*
- *If $\#\mathcal{A} = 3$, then one between $(\mathcal{A})_1, (\mathcal{A})_2$ is cancellative, and the other one is finite.*

Lemma 23

Let $\mathcal{A}, \mathcal{B}$ be MV-chains, where $\mathcal{A}$ is simple, and $\mathcal{B}$ is non-trivial. If there is a homomorphism k from $\mathcal{A}$ to $\mathcal{B}$, then $\mathcal{A} \overset{k}{\hookrightarrow} \mathcal{B}$.

Lemma 24

Let $\mathcal{A}, \mathcal{B}$ be two non-trivial BL-chains, where $(\mathcal{A})_0$ simple. If there is a homomorphism k from $\mathcal{A}$ to $\mathcal{B}$, then $(\mathcal{A})_0 \xhookrightarrow{k\restriction(\mathcal{A}_0)} (\mathcal{B})_0$ and $(\mathcal{A})_0 \overset{k}{\hookrightarrow} \mathcal{B}$.

back

Proof of the main theorem: some technical lemmas - 2

Lemma 25 ([AM03])

- *Let* $\bigoplus_{i\in I}\mathcal{A}_i$ *be a non-trivial BL-chain. Then* $\mathbf{ISP}_u(\bigoplus_{i\in I}\mathcal{A}_i) = \mathbf{I}(\bigoplus_{i\in I}\mathbf{SP}_u(\mathcal{A}_i))$*, where* $\bigoplus_{i\in I}\mathbf{SP}_u(\mathcal{A}_i) = \{\bigoplus_{i\in I}\mathcal{B}_i : \mathcal{B}_i \in \mathbf{SP}_u(\mathcal{A}_i)\}$.
- *If* $\mathcal{A}$ *is an infinite totally ordered cancellative hoop, then* $\mathbf{ISP}_u(\mathcal{A}) = Ch(\mathbb{CH})$.
- *If* $\mathcal{A}$ *is a totally ordered Wajsberg hoop with infinite rank, and for every* $n \geq 2$, $\mathcal{L}_n \hookrightarrow \mathcal{A}$*, then* $\mathbf{ISP}_u(\mathcal{A}) = Ch(\mathcal{A})$.
- *If* $\mathcal{A}$ *is a totally ordered Wajsberg hoop with* $rank(\mathcal{A}) = n$*, and* $\mathcal{L}_n \hookrightarrow \mathcal{A}$*, then* $\mathbf{ISP}_u(\mathcal{A}) = Ch(\mathcal{A})$*. If in addition* $\mathcal{A}$ *is also finite, then* $\mathbf{ISP}_u(\mathcal{A}) = \mathbf{IS}(\mathcal{A}) = Ch(\mathcal{A})$.

Lemma 26

Let S *be a finite set of BL-chains such that, for every* $\mathcal{A} \in S$.

- $\mathcal{A}$ *has finitely many components.*
- *Each* $(\mathcal{A})_i$ *is either cancellative or it is a Wajsberg hoop with finite rank such that* $(\mathcal{A})_i / Rad((\mathcal{A})_i) \hookrightarrow (\mathcal{A})_i$.

Let $\mathbb{L} = \mathbf{V}(S)$*. Then the following hold.*

1. $Ch(\mathbb{L}) = \mathbf{ISP_u}(S) = \bigcup_{\mathcal{T}\in S}\mathbf{ISP_u}(\mathcal{T})$.
2. *In particular, for every* $\mathcal{A} = \bigoplus_{i=0}^{k}\mathcal{A}_i \in S$ *such that* $(\mathcal{A})_0$ *is finite,* $Ch(\mathcal{A}) = \mathbf{I}(\mathbf{S}(\mathcal{A}_0) \oplus \bigoplus_{i=1}^{k}\mathbf{SP_u}(\mathcal{A}_i))$.
3. *If every* $\mathcal{A} \in S$ *is finite, then* $Ch(\mathbb{L}) = \mathbf{IS}(S) = \bigcup_{\mathcal{T}\in S}\mathbf{IS}(\mathcal{T})$.

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