# Prenuclear vs. nuclear objects in $*-$autonomous categories

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# Next

1. Recaps, motivations, and a conjecture

2. Frobenius structures

3. A partial solution of the conjecture

4. Questioning pseudoaffine

# Frobenius quantales

## Definition

A ***Frobenius quantale*** is a quantale $(Q, \star)$ coming with antitone "negations" ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q$ satisfying:

$$({}^{\perp}x)^{\perp} = {}^{\perp}(x^{\perp}) = x \qquad \text{(inverse to each other)}$$

$$x \multimap {}^{\perp}y = x^{\perp} \circ\!\!- y \qquad \text{(contraposition law)}$$

## Remarks

- A provability model of non-commutative classical linear logic.
- Similar to *involutive residuated lattice*, but complete.
- Such a quantale is until if and only if it has a dualizing element.
- The above axiomatization allows to consider Frobenius quantales with neither a unit nor a dualizing element.

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# Motivations

**Theorem (Egger, Kruml, Paseka ~ 2008, Santocanale 2020)**

*Let $L$ be a complete lattice. The following are equivalent:*

1. *The quantale $[L, L]_\vee$ of join-preserving endomaps of $L$ is Frobenius.*
2. *$L$ is a completely distributive lattice.*

**Theorem (Raney 1960, Higgs and Rowe 1989)**

*The nuclear objects in the category of complete sup-lattices are exactly the completely distributive lattice.*

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# Motivations

**Conjecture**

Let $A$ be an object of a SMCC. The following are equivalent:

1. The object $[A, A]$ of endomorphisms of $A$ has a Frobenius structure.
2. $A$ is nuclear.

**Theorem (Raney 1960, Higgs and Rowe 1989)**

*The nuclear objects of the category of complete sup-lattices are exactly the completely distributive lattices.*

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# Dual pairings

**Definition**

A triple $(A, B, \epsilon)$ is said to be a ***dual pairing*** (w.r.t. the object 0) if

$$\epsilon : A \otimes B \longrightarrow 0$$

and the two induced natural transformations are isomorphims.

$$\hom(X, B) \longrightarrow \hom(A \otimes X, 0)\,, \qquad \hom(X, A) \longrightarrow \hom(X \otimes B, 0)\,.$$

**Example**

- In a $*$-autonomous category

$$ev : A \otimes A^* \longrightarrow 0\,, \qquad \epsilon : (A \otimes A^*) \otimes [A, A] \longrightarrow 0$$

are dual pairings.

- In **SLatt**, the map

$$\epsilon(x, y) = \begin{cases} \bot\,, & x \leq y\,, \\ \top\,, & x \not\leq y\,, \end{cases}$$

yields a dual pairing $\epsilon_L : L \otimes L^{op} \longrightarrow \mathbf{2}$.

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# Frobenius quantales, once more

In a Frobenius quantale $(Q, \star, {}^{\perp}(-), (-)^{\perp})$, we have

- $(Q, Q^{\mathrm{op}}, \epsilon)$ is a dual pairing;
- $(Q, \star)$ is a semigroup;
- ${}^{\perp}(-), (-)^{\perp} : Q \to Q^{\mathrm{op}}$ and $x \leq {}^{\perp}y$ iff $y \leq x^{\perp}$;
- $y \multimap {}^{\perp}x = y^{\perp} \multimapinv x$

$$\begin{array}{ccc} Q \otimes Q & \xrightarrow{Q \otimes (-)^{\perp}} & Q \otimes Q^{\mathrm{op}} \\ {\scriptstyle {}^{\perp}(-) \otimes Q} \big\downarrow & & \big\downarrow {\scriptstyle \alpha^{\ell}} \\ Q^{\mathrm{op}} \otimes Q & \xrightarrow[\alpha^{\rho}]{} & Q^{\mathrm{op}}. \end{array}$$

# Frobenius structures

**Definition**

A ***Frobenius structure*** is a tuple $(A, B, \epsilon, \mu_A, l, r)$ where

- $(A, B, \epsilon)$ is a dual pairing;
- $(A, \mu_A)$ is a semigroup;
- $l, r : A \longrightarrow B$ and $(l, r)$ is an adjunction with both maps invertible

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$$\begin{array}{ccc} A \otimes A & \xrightarrow{A \otimes r} & A \otimes B \\ {\scriptstyle l \otimes A}\downarrow & & \downarrow{\scriptstyle \alpha^{\ell}} \\ B \otimes A & \xrightarrow[\alpha^{\rho}]{} & B. \end{array}$$

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# Nuclearity

From here, $\mathcal{V}$ is symmetric monoidal closed and $0 = I$.

## Definition

For every object $A$ of $\mathcal{C}$, there exists a canonical arrow

$$\mathtt{mix}_A : A^* \otimes A \longrightarrow [A, A].$$

An object $A$ is ***nuclear*** if $\mathtt{mix}_A$ is an isomorphism.

## Theorem (Raney 1960, Higgs and Rowe 1989)

*The nuclear objects of **SLatt** are exactly the completely distributive lattices.*

# From Nuclearity to Frobenius structure, and back

**Theorem (LS and CL, CSL 2023)**

If an object $A$ of $\mathcal{V}$ is nuclear, then $[A, A]$ can be endowed with a Frobenius structure.

If $\mathcal{V}$ is $*$-autonomous and $A$ is pseudoaffine, then the converse hold.

**Definition**

An object $A$ of $\mathcal{V}$, is ***pseudoaffine*** if (either is initial or) $I$ embeds into $A$ as a retract (*i.e* if there exists $p : I \to A$ and $c : A \to I$ such that $c \circ p = \mathrm{id}_I$.).

**Example**

Every object of **SLatt**, $k$-**Vect**, **Coh HypCoh**, is pseudoaffine.

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# Is pseudoaffine necessary ?

**Question:**

Can we have a $*$-autonomous category $\mathcal{V}$ and a object $A$ of $\mathcal{V}$ such that

- $A$ is not nuclear,
- $A$ is not pseudoaffine,
- $A$ is prenuclear, that is $[A, A]$ is Frobenius ?

**Definition (Schalk and de Paiva 2004 category $P$-Set)**

Let $(P, \leq)$ be a poset (the base category). We define the category $P$-**Set**:

- An object: a pair $(X, \alpha)$ with $X$ a set and $\alpha : X \to P$;
- An arrow $(X, \alpha) \to (Y, \beta)$: a relation $R \in P(X \times Y)$ such that

$$xRy \text{ implies } \alpha(x) \leq \beta(y).$$

**Theorem (Schalk and de Paiva 2004)**

*For $(Q, \star, 1)$ a unital commutative Frobenius quantale, $Q$-**Set** is $*$-autonomous.*

## Various characterisations

We take $Q$ such that $1 = 0$ in $Q$, so $I = 0$ in $Q$-**Set**.

**Lemmas**

A $Q$-**Set** $(X, \alpha)$ is

- pseudoaffine iff $\alpha(x) = 1$, for some $x \in X$,
- nuclear iff,

$$\alpha(x) \multimap \alpha(y) = \alpha(x)^{\perp} \star \alpha(y) \qquad (\forall x, y)$$

  Equivalently: iff $\alpha(x)$ is invertible $(\forall x)$

- prenuclear iff, for a pair of inverse maps $(f, g)$ on $X$,

$$\alpha(x) \multimap \alpha(y) = \alpha(fx)^{\perp} \star \alpha(y) = \alpha(x)^{\perp} \star \alpha(gy) \qquad (\forall x, y)$$

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# Finding the right $Q$

- Take $\mathbb{Z}$.
- Add $\pm\infty$.
- Add a new unit $u$ such that $0 < u < 1$.

Then $\mathbb{Z} \subseteq Q$ is prenuclear, with $f(x) = x - 1$ and $g(x) - x + 1$.

# Discussing the pseudoaffine condition

We take $Q$ such that $1 = 0$ in $Q$, so $I = 0$ in $Q$-**Set**.

**Theorem (LS and CL)**

If $(X, \alpha)$ is prenuclear and any of the following conditions holds:

- the quantale $Q$ has no infinite chain,
- $X$ is finite,
- $\alpha(x)$ is invertible, for some $x \in X$,
- $(X, \alpha)$ is Girard, that is, $f = g$,

then $(X, \alpha)$ is nuclear.

**Theorem (LS and CL)**

There is a quantale $Q$ and an object $(X, \alpha)$ of $Q$-**Set** which is prenuclear but not nuclear.

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# Conclusions

## Results

- A definition of Frobenius structures in autonomous categories.
- Proof of our conjecture up to a technical (but quite natural) hypothesis (pseudoaffine).
- Other results,e.g. an abstraction of the double negation construction.
- Testing the pseudoaffine condition.

## To do next

- What about Girard quantales/structures and nuclearity ?
- Understand "how much" we need $*$-autonomous categories.
- Monoidal fibrations.

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Thank you!

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