The Structure of Locally Integral Involutive PO-Monoids and Semirings

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Joint work with Peter Jipsen and Siddhartha Lodhia

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An **involutive partially ordered monoid**, or **ipo-monoid** for short, is a structure $(A, \leqslant, \cdot, 1, \sim, -)$ such that

- $(A, \leqslant)$ is a poset,
- $(A, \cdot, 1)$ is a monoid,
- $x \leqslant y \iff x \cdot {\sim}y \leqslant 0 \iff -y \cdot x \leqslant 0$ (ineg),

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where $0 = -1 = {\sim}1$.

Some examples: **all groups** (if $\leqslant$ is $=$), **all partially ordered groups** (where ${\sim}x = -x = x^{-1}$), MV-algebras are ipo-monoids ($\vee, \wedge$ are definable).

Some Properties of ipo-monoids

double negation:	${\sim}{-}x = x = -{\sim}x$
rotation:	$x \cdot y \leqslant z \iff y \cdot {\sim}z \leqslant {\sim}x \iff -z \cdot x \leqslant -y$
antitonicity:	$x \leqslant y \iff {\sim}y \leqslant {\sim}x \iff -y \leqslant -x$
residuation (res):	$x \cdot y \leqslant z \iff x \leqslant -(y{\cdot}{\sim}z) \iff y \leqslant {\sim}(-z{\cdot}x)$
constants:	$0 = {\sim}1, \quad {\sim}0 = 1 \quad \text{and} \quad -0 = 1.$

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Ipo-monoids satisfy: $x \backslash -y = \sim\!x/y$. (So Frobenius quantales are ipo-monoids.)

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- $-x \cdot x = x \cdot {\sim}x$,
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Proposition

Every integral ipo-monoid is locally integral.

Let $A^+ = \{x \in A \mid 1 \leqslant x\}$ be the **positive cone** of A.

Notice

$$1 \cdot x = x \quad \Longrightarrow \quad 1 \cdot x \leqslant x \quad \Longrightarrow \quad 1 \leqslant x/x.$$

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These terms are *important* enough to introduce *special notation*:

$$1_x = x/x = x\backslash x \quad \text{and} \quad 0_x = {\sim}1_x = -1_x$$

We also have $1_x = {\sim}0_x = -0_x$. Moreover, $1_x, 0_x$ are idempotent.

The following properties hold in every *locally integral* ipo-monoid **A**:

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- $y \in [0_x, 1_x] \iff [0_y, 1_y] \subseteq [0_x, 1_x]$,
- $y \in A_x \iff y \in [0_x, 1_x]$ and $1_x \cdot y = y$.

$[0_x, 1_x] = \{a \in A\colon 0_x \leqslant a \leqslant 1_x\}$ and $A_x = \{y \in A : 1_x = 1_y\}$.

Canonical Representatives for the A_x Equivalence Classes

$x \equiv y \iff 1_x = 1_y$ is an **equivalence relation**.

Hence the equivalence classes $\{A_x : x \in A\}$ *partition* A.

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Lemma

Let $\mathbf{A}$ *be a locally integral ipo-monoid, and* $p, a, x \in A$*.*

- $p \in A^+ \iff p = 1_p$.
- $a \in {\downarrow}0 \iff a = 0_a$.
- $A_x \cap A^+ = \{1_x\}$.
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It follows that the classes A_x are **indexed** by the elements of A^+.

The Integral Components of Locally Integral ipo-monoids

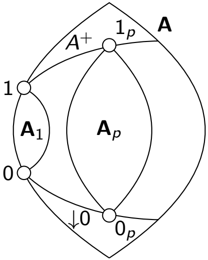
Theorem

Let $\mathbf{A}$ *be a locally integral ipo-monoid. For every* p *in* A^+*,*

- A_p *is closed under* $\sim, -, \cdot$*,*
- $\mathbf{A}_p = (A_p, \leqslant, \cdot, 1_p, \sim, -)$ *is an integral ipo-monoid,*
- *if* $(A, \leqslant)$ *is a lattice then* $(A_p, \leqslant, 0_p, 1_p)$ *is a bounded lattice.*

The structures $\mathbf{A}_p$ are called the **integral components** of $\mathbf{A}$.

Representation of the structure of a locally integral ipo-monoid



Proposition

All positive elements of a locally integral ipo-monoid $\mathbf{A}$ *are central,*

$$\text{for every } p \in A^+ \text{ and every } x \in A, \quad p \cdot x = x \cdot p.$$

Lemma

If $\mathbf{A}$ *is a locally integral,* $p, q \in A^+$,

$$x \in A_p \text{ and } y \in A_q \qquad \Longrightarrow \qquad x \cdot y \in A_{pq}.$$

Moreover, $1_p \cdot 1_q = 1_{pq}$ *and* $0_p \cdot 0_q = 0_{pq}$.

Products Between Components

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By duality, $a, b \leqslant 0$ implies $a \cdot b = a \wedge b$.

A family $\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$ of homomorphisms is **compatible** if

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If $\{\varphi_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j : i \leqslant j\}$ is indexed by the order of a lower-bounded join-semilattice $(I, \vee, \perp)$,
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- $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$, σ n-ary, $j = i_1 \vee \cdots \vee i_n$,

$$\sigma^{\mathbf{S}}(a_1, \ldots, a_n) = \sigma^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n)),$$

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- if σ is constant, $\sigma^{\mathbf{S}} = c^{\mathbf{A}_\bot}$.

Compatible Maps Between Integral Components

For positive $p \leqslant q$, **define** $\varphi_{pq} \colon A_p \to A_q$ by $\varphi_{pq}(x) = q \cdot x$.

Proposition

Let $\mathbf{A}$ *be a locally integral ipo-monoid and* $p \leqslant q$ *positive.*

- $\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q$ *is a monoidal homomorphism,*

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- $\{\varphi_{pq} : p \leqslant q\}$ *is a compatible family of monoidal homomorphisms.*

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Then $\left(\biguplus A_p, \leqslant^{\mathbf{S}}, \cdot^{\mathbf{S}}, {\sim}^{\mathbf{S}}, -^{\mathbf{S}}\right) = \mathbf{A}$.

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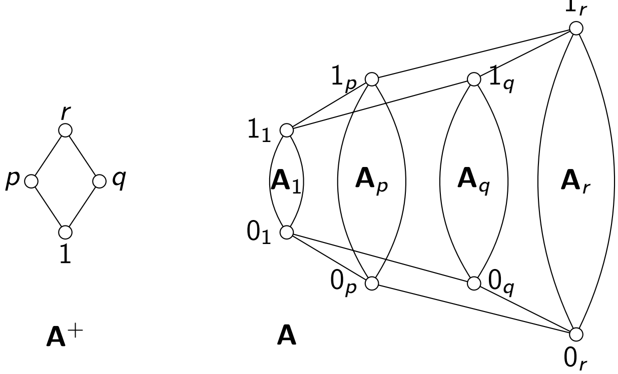
Moreover, if $\mathbf{A}$ *is in InRL then all* $\mathbf{A}_p$ *are in InRL.*

Furthermore, $\mathbf{A}$ *is commutative if and only if all its components are commutative*

A Generic Example with 4 Integral Components



Let $(D, \vee, 1)$ be a lower-bounded join-semilattice;
$\mathbf{A}_p = (A_p, \leqslant_p, \cdot_p, 1_p, \sim_p, -_p)$ integral ipo-monoid, for every $p \in D$;
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- $\sim^G a = \sim_p a$ and $-^G a = -_p a$,
- $a \leqslant^G b \iff a \cdot^G \sim^G b = 0_{p \vee q}$.

Let $(D, \vee, 1)$ be a lower-bounded join-semilattice;
$\mathbf{A}_p = (A_p, \leqslant_p, \cdot_p, 1_p, \sim_p, -_p)$ integral ipo-monoid, for every $p \in D$;
$\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^D q\}$ compat. family of monoidal hom.

Define the structure:

$$\int_\Phi \mathbf{A}_p = \left(\biguplus_D A_p, \leqslant^G, \cdot^G, 1^G, \sim^G, -^G \right)$$

where $\left(\biguplus_D A_p, \cdot^G, 1^G \right)$ is the Płonka sum of the family Φ
and for all $p, q \in D$, $a \in A_p$, and $b \in A_q$,

- $\sim^G a = \sim_p a$ and $-^G a = -_p a$,
- $a \leqslant^G b \iff a \cdot^G \sim^G b = 0_{p \vee q}$.

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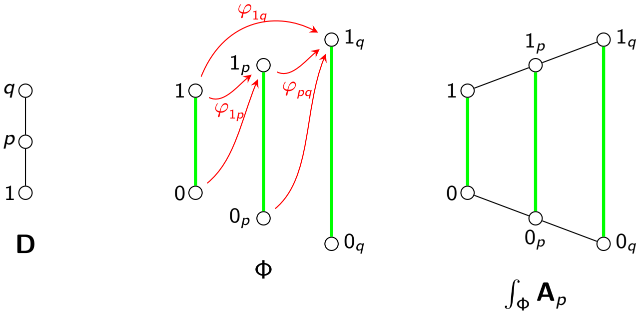
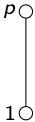
$\int_\Phi \mathbf{A}_p$ is the **glueing of** $\{\mathbf{A}_p : p \in D\}$ **along the family** Φ.

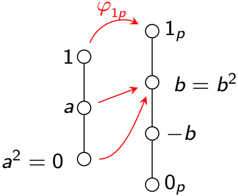
A Sugihara Glueing of Copies of the Standard MV-chain



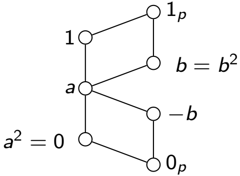
Glueing Ł$_3$ into a Small IMTL-algebra



$\mathbf{D}$

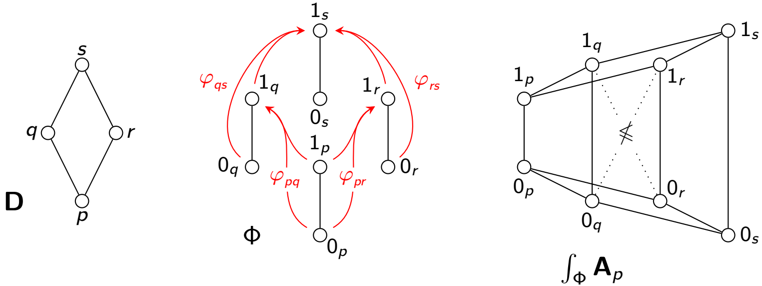


Φ



$\int_\Phi \mathbf{A}_p$

Glueing of Integral ipo-monoids that is not an ipo-monoid



The relation $\leqslant$ of $\int_\Phi \mathbf{A}_p$ is not transitive.

Required Conditions for Glueing Integral ipo-monoids

(balanced): for all $p, q \in D, a \in A_p, b \in A_q$,

$$a \cdot^G \sim^G b = 0_{p \vee q} \iff -^G b \cdot^G a = 0_{p \vee q}.$$

(zero): for all $p \leqslant^D q, \quad \varphi_{pq}(0_p) = 0_q \iff p = q.$

(tr): for all $a, b, c \in \biguplus A_p, \quad$ if $a \leqslant^G b$ and $b \leqslant^G c$, then $a \leqslant^G c$.

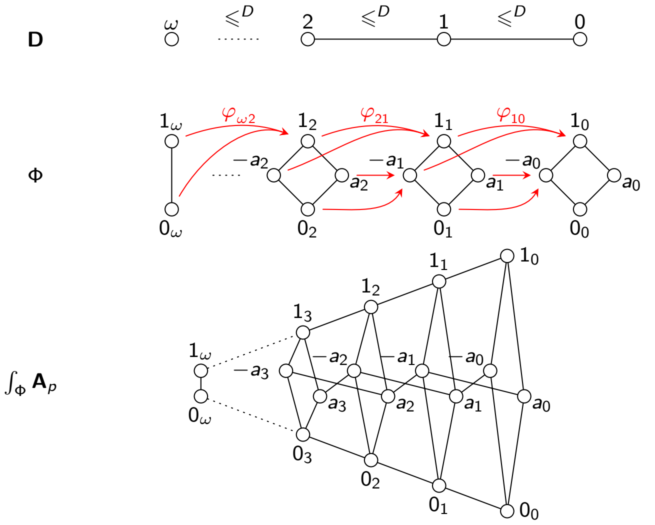
Theorem

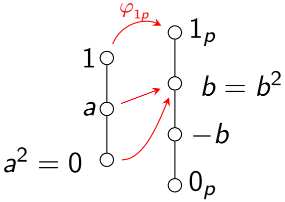
A structure $\mathbf{A}$ is a locally integral ipo-monoid if and only if there is

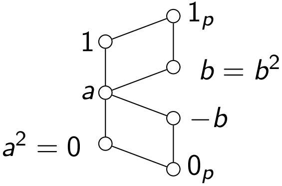
- *a lower-bounded join-semilattice $\mathbf{D}$,*
- *a family of integral ipo-monoids $\{\mathbf{A}_p : p \in D\}$, and*
- *a compatible family $\Phi = \{\varphi_{pq} \colon \mathbf{A}_p \to \mathbf{A}_q : p \leqslant^{\mathbf{D}} q\}$ of monoidal homomorphisms satisfying (bal), (zero), and (tr)*

so that $\mathbf{A} = \int_\Phi \mathbf{A}_p$.

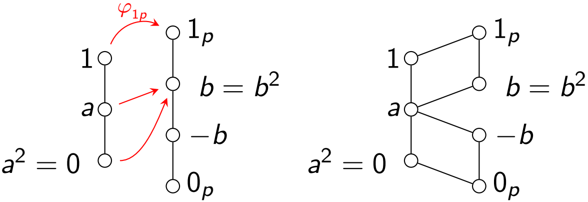
Glueing of infinitely many Boolean algebras that produces an involutive semiring

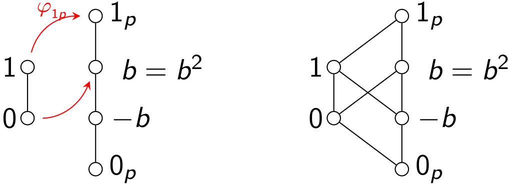



φ_{1_p}
1_p
1
$b = b^2$
a
$-b$
$a^2 = 0$
0_p


1_p
1
$b = b^2$
a
$-b$
$a^2 = 0$
0_p

Two glueings, one being a semiring, the other no





The condition (tr) can be replaced by more "local" condition.

- for all $p \leqslant^{\mathbf{D}} q$, and $a, b \in A_p$, $a \leqslant_p b \implies \varphi_{pq}(a) \leqslant_q \varphi_{pq}(b)$; (mon)
- for all $p \leqslant^{\mathbf{D}} q$, $p \leqslant^{\mathbf{D}} r$, and $a \in A_p$, $\sim\varphi_{pq}(a) \leqslant^{\mathbf{G}} \varphi_{pr}(\sim a)$; (lax)
- for all $p \vee r \leqslant^{\mathbf{D}} v$, $a \in A_p$, and $b \in A_r$,

$$\varphi_{rv}(\sim b) \leqslant_v \sim\varphi_{pv}(a) \implies a \leqslant^{\mathbf{G}} b. \qquad (\sim\text{lax})$$

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A locally integral ipo-monoid **A** is idempotent if and only if all its integral components are Boolean algebras.

A Few Remarks and Questions

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Under which conditions **A** is lattice-ordered?

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Are locally integral ipo-monoids or InRLs decidable?

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THANKS!