# Algebra and Logic in Granularity

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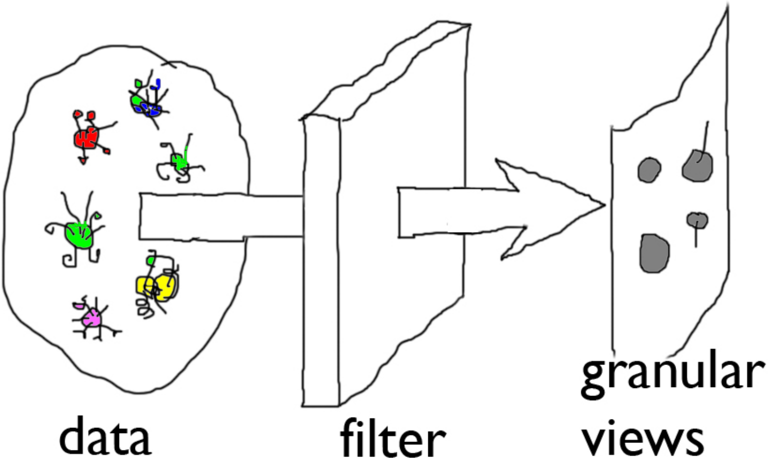
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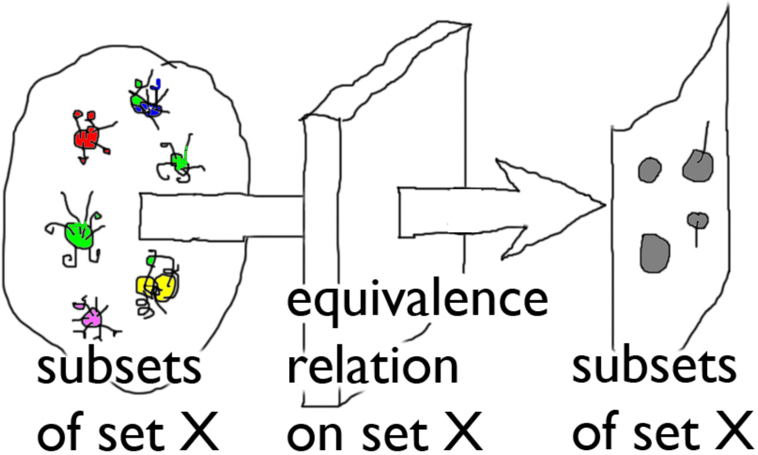
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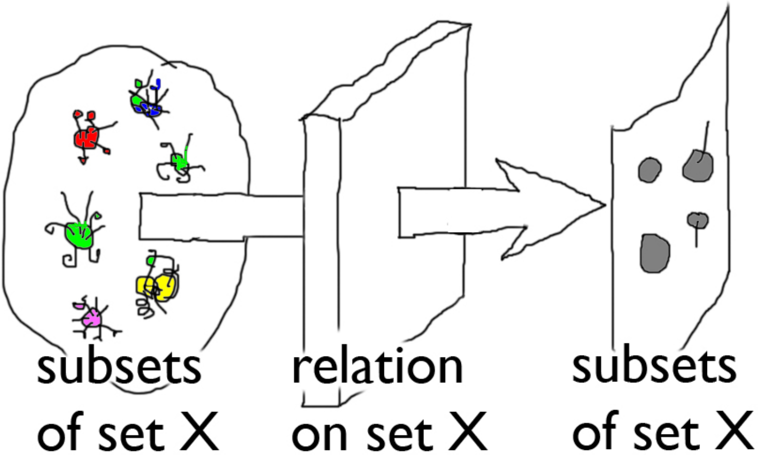
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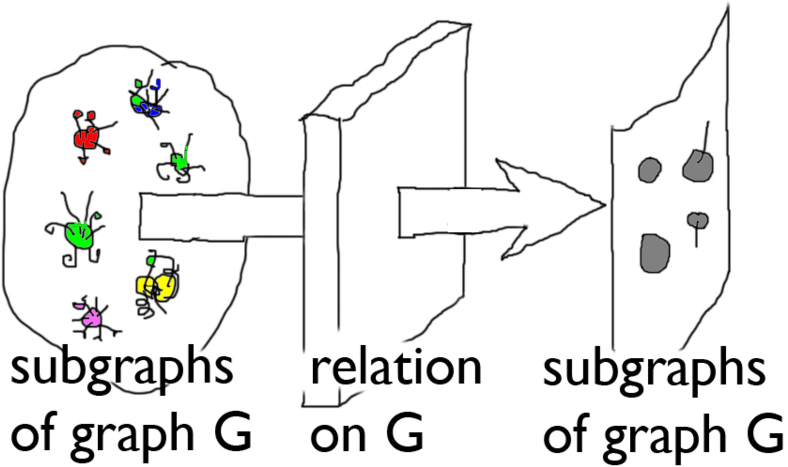
subgraphs of graph G
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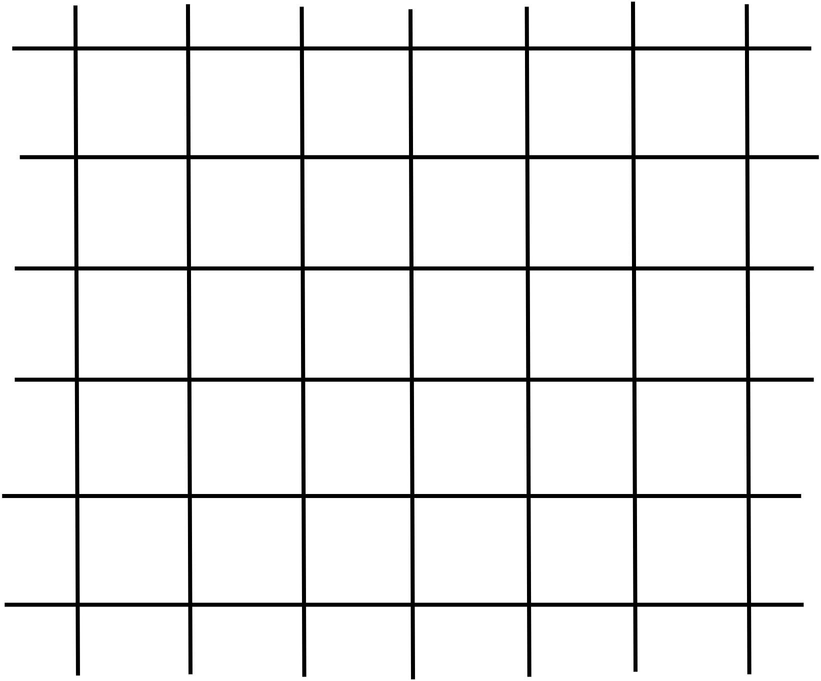
# (Cousty, Najman, Dias, Serra, 2013)

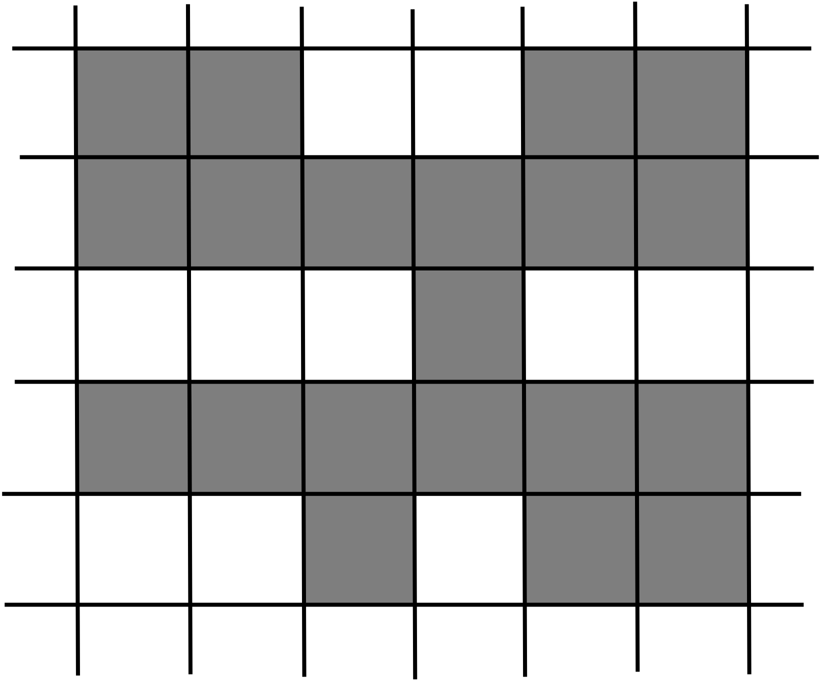
(a) original binary image

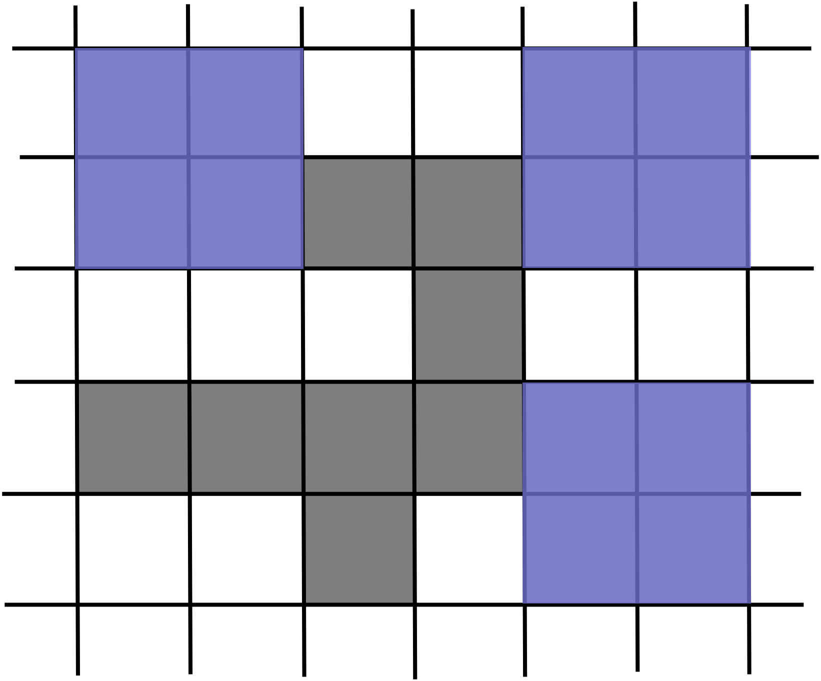
(b) noisy binary image

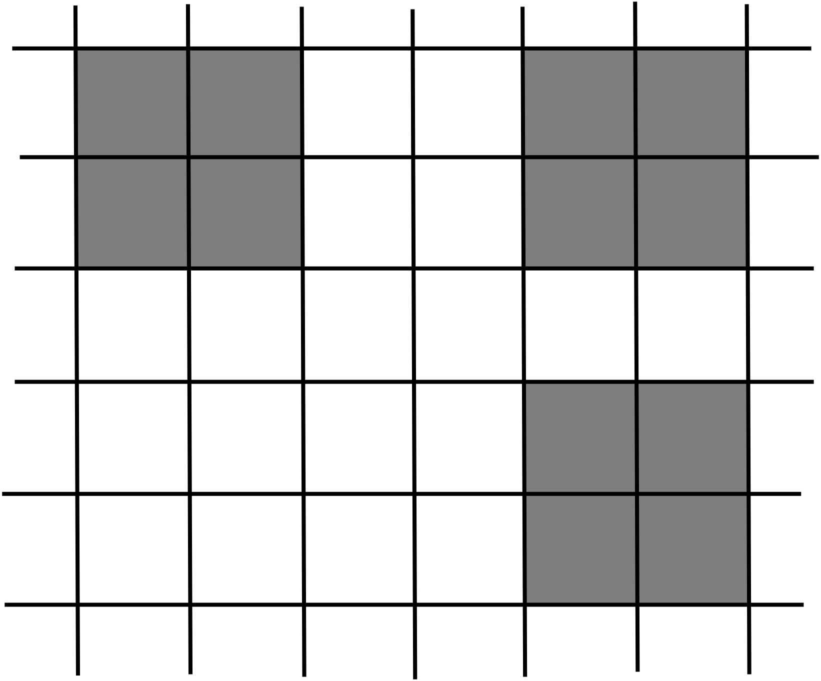
(c) usual ASF

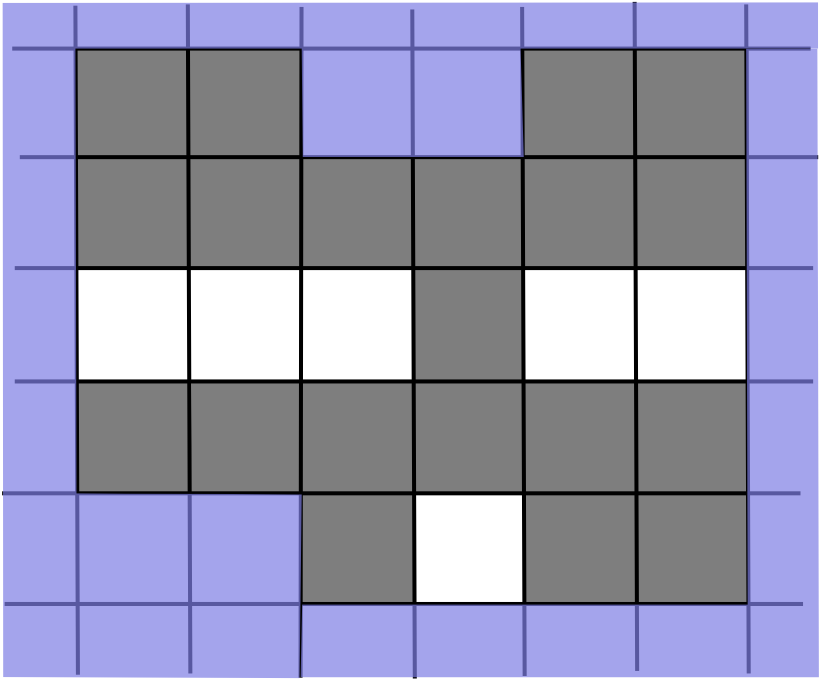
(d) graph ASF

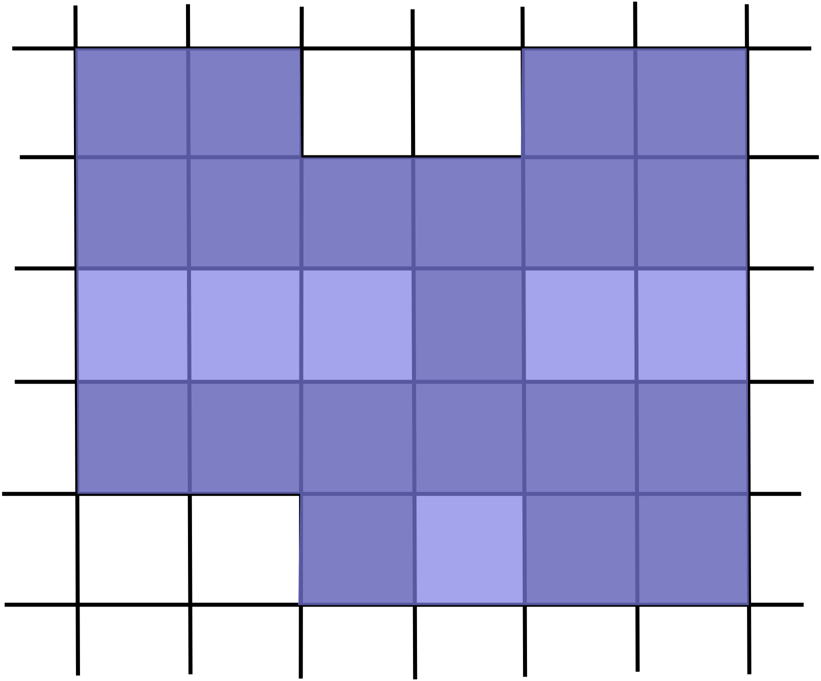


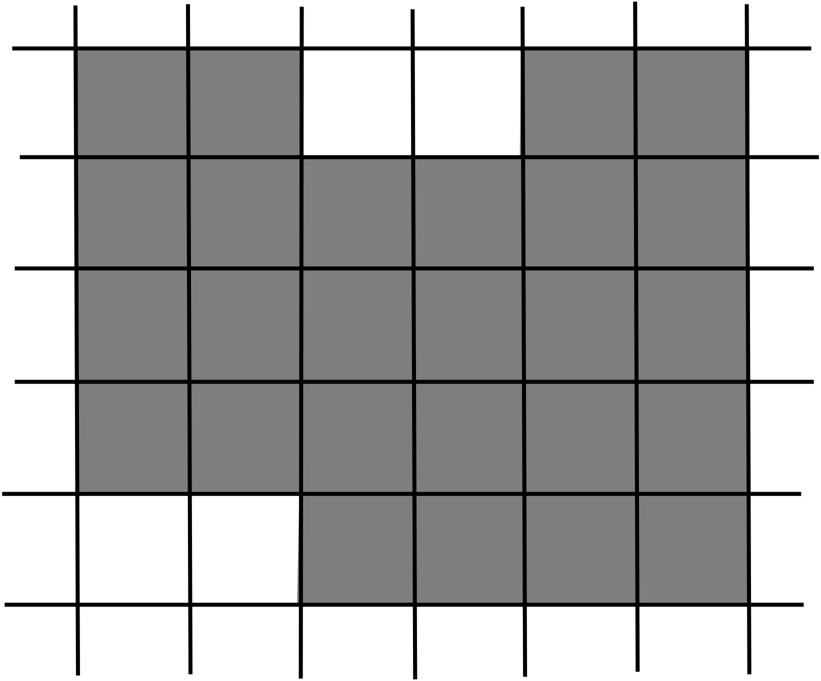


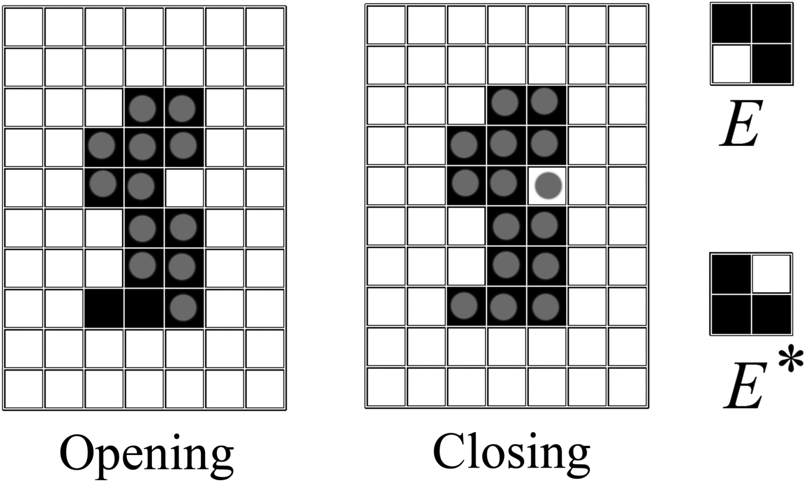








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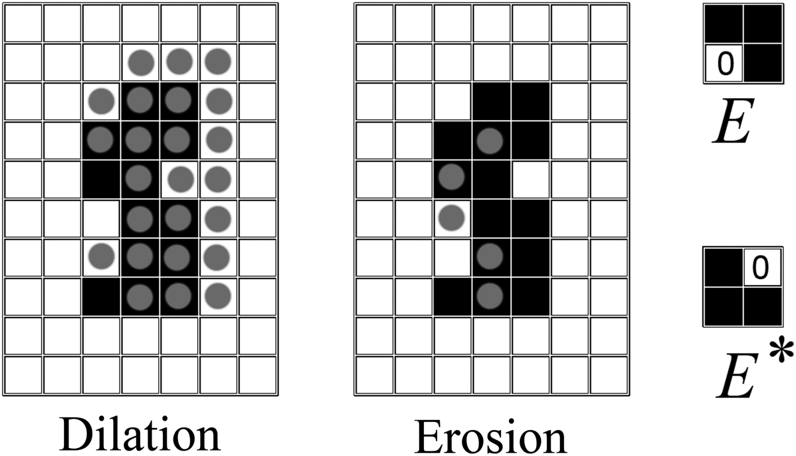

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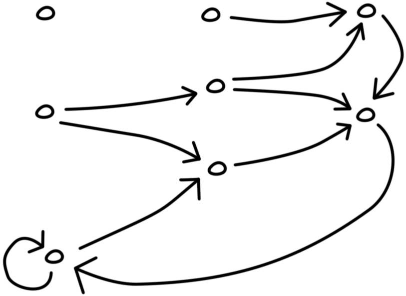

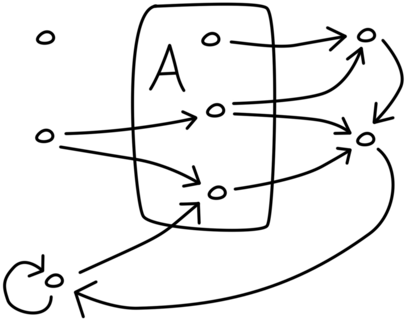
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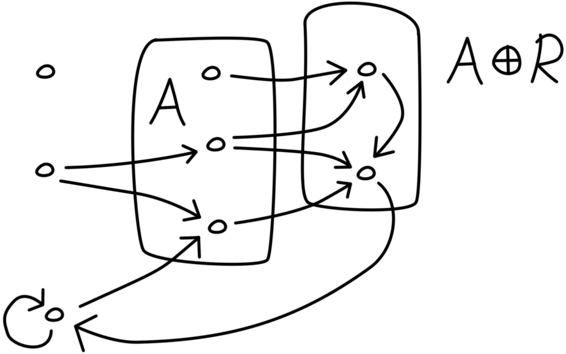
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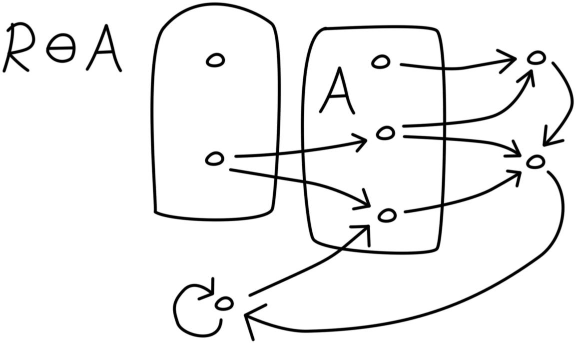

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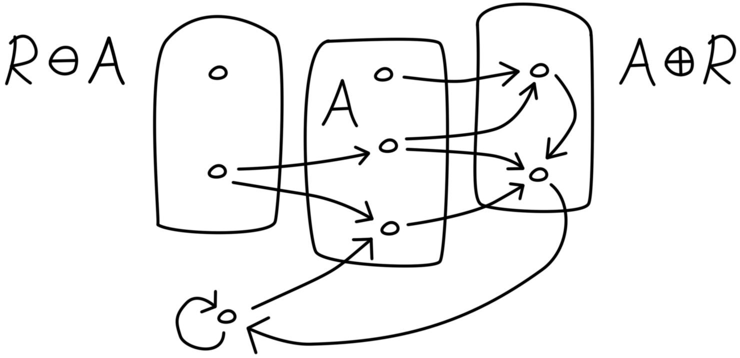
A


R⊖A
A
A⊕R

$$2^X \underset{R \ominus \_}{\overset{\_ \oplus R}{\rightleftarrows}} 2^X \qquad 2^X \underset{\ulcorner R \urcorner}{\overset{\llcorner R \lrcorner}{\rightleftarrows}} 2^X$$

$$\llcorner R \lrcorner\, A \;=\; \{x \in X \mid \exists b\, (b\; R\; x \wedge b \in A)\}$$

$$\ulcorner R \urcorner\, A \;=\; \{x \in X \mid \forall b\, (x\; R\; b \rightarrow b \in A)\}$$

## Definition

*Let $Q$ be a quantale. A* **right $Q$-module** *is a sup-lattice $M$ equipped with an action $\oplus : M \times Q \to M$ such that*

1. $m \oplus 1 = m$,
2. $m \oplus (p \cdot q) = (m \oplus p) \oplus q$,
3. $m \oplus \bigvee A = \bigvee \{m \oplus a \mid a \in A\}$,
4. $\bigvee B \oplus q = \bigvee \{b \oplus q \mid b \in B\}$.

Key example:

- $M$ is $2^X$
- $Q$ is $2^{X \times X}$
- $\cdot$ is composition of relations

Given a right $Q$-module, $\oplus : M \times Q \to M$,

fix $q \in Q$ and consider $m \mapsto m \oplus q$ from $M$ to $M$.

This is sup-preserving, so has a right adjoint: $q \ominus \_ : M \to M$.

**Lemma**

$\ominus : Q \times M \to M$ *satisfies*

1. $1 \ominus m = m$,
2. $(p \cdot q) \ominus m = p \ominus (q \ominus m)$,
3. $q \ominus \bigwedge B = \bigwedge\{q \ominus b \mid b \in B\}$,
4. $\bigvee A \ominus m = \bigwedge\{a \ominus m \mid a \in A\}$.

$\ominus : Q \times M^{\mathsf{op}} \to M^{\mathsf{op}}$ is a left $Q$-module

Let

- $\varphi : M \to M^{\mathsf{op}}$ be a sup-lattice isomorphism, and
- $\oplus : M \times Q \to M$ a right $Q$-module with
- adjoint left module $\ominus : Q \times M^{\mathsf{op}} \to M^{\mathsf{op}}$

Define $\oplus^{\varphi} : Q \times M \to M$ by $q \oplus^{\varphi} m = \varphi(q \ominus \varphi^{-1} m)$

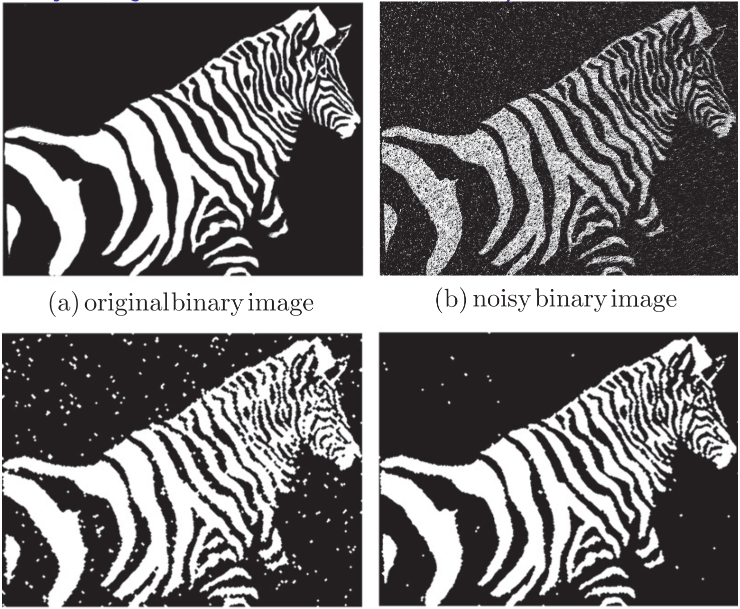
Define $\ominus^{\varphi} : M^{\mathsf{op}} \times Q \to M^{\mathsf{op}}$ by $m \ominus^{\varphi} q = \varphi^{-1}(\varphi m \oplus q)$

Then $\oplus^{\varphi}$ is a left $Q$-module and $\ominus^{\varphi}$ is its adjoint right module

Key example:

- $\varphi$ is complementation on subsets
- $\oplus^{\varphi}$ is dilation by converse

# (Cousty, Najman, Dias, Serra, 2013)



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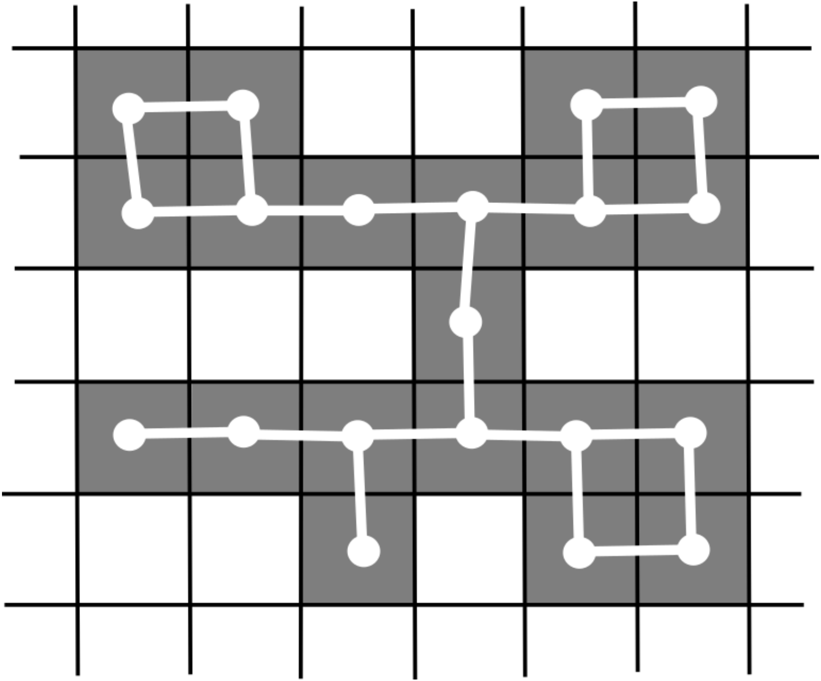


| Operator | Applied to $\mathbb{G}$ |
|---|---|
| $[\delta, \Delta]$ | $(\delta(\mathbb{G}^\bullet), \Delta(\mathbb{G}^\times))$ |
| $[\epsilon, \mathcal{E}]$ | $(\epsilon(\mathbb{G}^\bullet), \mathcal{E}(\mathbb{G}^\times))$ |
| $[\gamma, \Gamma]_1$ | $(\gamma_1(\mathbb{G}^\bullet), \Gamma_1(\mathbb{G}^\times))$ |
| $[\varphi, \Phi]_1$ | $(\varphi_1(\mathbb{G}^\bullet), \Phi_1(\mathbb{G}^\times))$ |
| $[\gamma, \Gamma]_{1/2}$ | $(\gamma_{1/2}(\mathbb{G}^\bullet), \Gamma_{1/2}(\mathbb{G}^\times))$ |
| $[\varphi, \Phi]_{1/2}$ | $(\varphi_{1/2}(\mathbb{G}^\bullet), \Phi_{1/2}(\mathbb{G}^\times))$ |
| $[\gamma, \Gamma]_{(2i+j)/2}$ | $[\delta, \Delta]^i \, ([\gamma, \Gamma]_{1/2})^j \, [\epsilon, \mathcal{E}]^i \, \mathbb{G}$ |
| $[\varphi, \Phi]_{(2i+j)/2}$ | $[\epsilon, \mathcal{E}]^i \, ([\varphi, \Phi]_{1/2})^j \, [\delta, \Delta]^i \, \mathbb{G}$ |

$i \in \mathbb{N}$ and $j \in \{0, 1\}$.

# filters

In mathematical morphology, given a complete lattice $\mathcal{L}$, the term 'filter' means any function $f : \mathcal{L} \to \mathcal{L}$ which is

- order-preserving, and
- idempotent.

Examples of filters include

- openings (which also satisfy $fx \leqslant x$) and
- closings (which also satisfy $x \leqslant fx$).

Consider sequences of filters $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ where,

- $\alpha_1 \leqslant \beta_1$ and
- for $\lambda \leqslant \mu$, $\quad \alpha_\mu \leqslant \alpha_\lambda$
- for $\lambda \leqslant \mu$, $\quad \beta_\lambda \leqslant \beta_\mu$

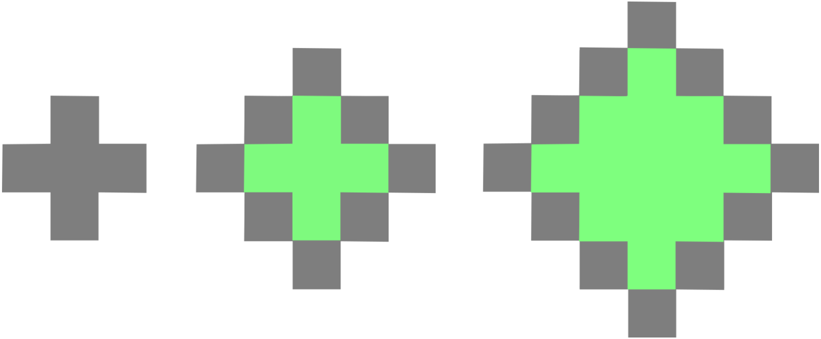
Composites of the forms

$$\alpha_\lambda \beta_\lambda \alpha_{\lambda-1} \beta_{\lambda-1} \cdots \alpha_1 \beta_1,$$
$$\beta_\lambda \alpha_\lambda \beta_{\lambda-1} \alpha_{\lambda-1} \cdots \beta_1 \alpha_1$$

are filters called 'alternating sequential filters'.

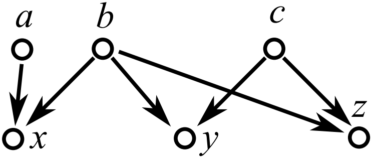
# Example

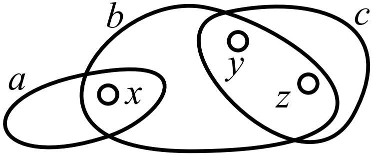
$\alpha_i, \beta_i$ are opening and closing by structuring element of size $i$



$\alpha_\lambda \beta_\lambda \alpha_{\lambda-1} \beta_{\lambda-1} \cdots \alpha_1 \beta_1$ $\qquad$ $\beta_\lambda \alpha_\lambda \beta_{\lambda-1} \alpha_{\lambda-1} \cdots \beta_1 \alpha_1$

# Hypergraph $\mathbb{H}$ alias relation $J \subseteq \mathbb{H}^{\times} \times \mathbb{H}^{\bullet}$




$$\mathbb{H}^{\bullet} = \{x, y, x\}, \qquad \mathbb{H}^{\times} = \{a, b, c\},$$

$$J = \{(a, x), (b, x), (b, y), (b, z), (c, y), (c, z)\}$$

Arbitrary relations $R : \mathbb{H}_1 \nrightarrow \mathbb{H}_2$ are unions of four:

$$R^{\bullet\bullet} \subseteq \mathbb{H}_1^{\bullet} \times \mathbb{H}_2^{\bullet}, \quad R^{\times\bullet} \subseteq \mathbb{H}_1^{\times} \times \mathbb{H}_2^{\bullet},$$
$$R^{\bullet\times} \subseteq \mathbb{H}_1^{\bullet} \times \mathbb{H}_2^{\times}, \quad R^{\times\times} \subseteq \mathbb{H}_1^{\times} \times \mathbb{H}_2^{\times}.$$

Write as matrix $R = \begin{bmatrix} R^{\bullet\bullet} & R^{\times\bullet} \\ R^{\bullet\times} & R^{\times\times} \end{bmatrix}$.

$1_{\mathbb{H}} = \begin{bmatrix} I^{\bullet} & J \\ \varnothing & I^{\times} \end{bmatrix}$ is a partial order on $X = \mathbb{H}^{\bullet} \cup \mathbb{H}^{\times}$

$R : X^{\mathsf{op}} \times X \to 2$ monotone iff $1_{\mathbb{H}}\, R\, 1_{\mathbb{H}} = R$

$X^{\mathsf{op}}$ represents the dual hypergraph $\mathbb{H}^d$

$$\mathbb{T} : \mathbb{H} \nrightarrow \mathbb{H}^d$$

$$\mathbb{T} = \begin{bmatrix} \breve{J} & \varnothing \\ \varnothing & J \end{bmatrix}$$

| Operator | Applied to $\mathbb{G}$ | Relational form |
|---|---|---|
| $[\delta, \Delta]$ | $(\delta(\mathbb{G}^\bullet), \Delta(\mathbb{G}^\times))$ | $\llcorner \breve{\mathbb{T}}\,\mathbb{T} \lrcorner$ |
| $[\epsilon, \mathcal{E}]$ | $(\epsilon(\mathbb{G}^\bullet), \mathcal{E}(\mathbb{G}^\times))$ | $\ulcorner \breve{\mathbb{T}}\,\mathbb{T} \urcorner$ |
| $[\gamma, \Gamma]_1$ | $(\gamma_1(\mathbb{G}^\bullet), \Gamma_1(\mathbb{G}^\times))$ | $\llcorner \breve{\mathbb{T}}\,\mathbb{T} \lrcorner \ulcorner \breve{\mathbb{T}}\,\mathbb{T} \urcorner$ |
| $[\varphi, \Phi]_1$ | $(\varphi_1(\mathbb{G}^\bullet), \Phi_1(\mathbb{G}^\times))$ | $\ulcorner \breve{\mathbb{T}}\,\mathbb{T} \urcorner \llcorner \breve{\mathbb{T}}\,\mathbb{T} \lrcorner$ |
| $[\gamma, \Gamma]_{1/2}$ | $(\gamma_{1/2}(\mathbb{G}^\bullet), \Gamma_{1/2}(\mathbb{G}^\times))$ | $\llcorner \breve{\mathbb{T}} \lrcorner \ulcorner \breve{\mathbb{T}} \urcorner$ |
| $[\varphi, \Phi]_{1/2}$ | $(\varphi_{1/2}(\mathbb{G}^\bullet), \Phi_{1/2}(\mathbb{G}^\times))$ | $\ulcorner \mathbb{T} \urcorner \llcorner \mathbb{T} \lrcorner$ |
| $[\gamma, \Gamma]_{(2i+j)/2}$ | $[\delta, \Delta]^i \, ([\gamma, \Gamma]_{1/2})^j \, [\epsilon, \mathcal{E}]^i \, \mathbb{G}$ | $\llcorner (\breve{\mathbb{T}}\,\mathbb{T})^i \breve{\mathbb{T}}^j \lrcorner \ulcorner (\breve{\mathbb{T}}\,\mathbb{T})^i \breve{\mathbb{T}}^j \urcorner$ |
| $[\varphi, \Phi]_{(2i+j)/2}$ | $[\epsilon, \mathcal{E}]^i \, ([\varphi, \Phi]_{1/2})^j \, [\delta, \Delta]^i \, \mathbb{G}$ | $\ulcorner \mathbb{T}^j (\breve{\mathbb{T}}\,\mathbb{T})^i \urcorner \llcorner \mathbb{T}^j (\breve{\mathbb{T}}\,\mathbb{T})^i \lrcorner$ |

In the last two rows $i \in \mathbb{N}$ and $j \in \{0, 1\}$.

Given poset $(X, \leqslant)$ the monotone relations $X^{\mathsf{op}} \times X \to 2$ form a quantale $Q$ which acts on the lattice $M$ of up-sets of $X$ by dilation.

The lattice of up-sets has

$$M^{\mathsf{op}} \underset{\lrcorner}{\overset{\lrcorner}{\rightleftarrows}} M \underset{\neg}{\overset{\neg}{\rightleftarrows}} M^{\mathsf{op}}$$

To Modify the action using these we need lax modules, which preserve sups but only satisfy (in right / left case)

1. $m \leqslant m \oplus 1$,
2. $m \oplus (p \cdot q) \leqslant (m \oplus p) \oplus q$,

1. $m \leqslant 1 \oplus m$,
2. $(p \cdot q) \oplus m \leqslant p \oplus (q \oplus m)$,

Analogous to
defining $\oplus^{\varphi} : Q \times M \to M$ by $q \oplus^{\varphi} m = \varphi(q \ominus \varphi^{-1} m)$

Don't get converse
$R : (1_{\mathbb{H}})^{\mathsf{op}} \times 1_{\mathbb{H}} \to 2$ is monotone
iff
$\breve{R} : 1_{\mathbb{H}} \times (1_{\mathbb{H}})^{\mathsf{op}} \to 2$ is monotone

but

$$\smile R = 1_{\mathbb{H}} \, \breve{R} \, 1_{\mathbb{H}}$$

# Bi-Intuitionisitic Modal Logic from Monotone Relations

$$
\begin{array}{rclcrcl}
[\![\bot]\!] & = & \varnothing & & [\![\top]\!] & = & X \\
[\![\varphi \vee \psi]\!] & = & [\![\varphi]\!] \cup [\![\psi]\!] & & [\![\varphi \wedge \psi]\!] & = & [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\neg\varphi]\!] & = & \ulcorner 1_{\mathbb{H}} \urcorner (-[\![\varphi]\!]) & & [\![\,\lrcorner\varphi]\!] & = & \llcorner 1_{\mathbb{H}} \lrcorner (-[\![\varphi]\!]) \\
[\![\varphi \to \psi]\!] & = & \ulcorner 1_{\mathbb{H}} \urcorner ((-[\![\varphi]\!]) \cup [\![\psi]\!]) & & [\![\varphi \succ \psi]\!] & = & \llcorner 1_{\mathbb{H}} \lrcorner ([\![\varphi]\!] \cap (-[\![\psi]\!])) \\
[\![\Box\varphi]\!] & = & \ulcorner R \urcorner [\![\varphi]\!] & & [\![\Diamond\varphi]\!] & = & \llcorner \smallsmile R \lrcorner [\![\varphi]\!] \\
[\![\blacklozenge\varphi]\!] & = & \llcorner R \lrcorner [\![\varphi]\!] & & [\![\blacksquare\varphi]\!] & = & \ulcorner \smallsmile R \urcorner [\![\varphi]\!]
\end{array}
$$

$$
[\![\Diamond\,\varphi]\!] = [\![\,\lrcorner\Box\neg\varphi]\!], \qquad [\![\blacksquare\,\varphi]\!] = [\![\neg\,\blacklozenge\,\lrcorner\varphi]\!]
$$

Relations $X \times X \to 2$ correspond to elements of $[2^X, 2^X]_\vee$

For a complete lattice $\Omega$

relations $X \times X \to [\Omega, \Omega]_\vee$ correspond to elements of $[\Omega^X, \Omega^X]_\vee$

Given $q, q' : X \times X \to [\Omega, \Omega]_\vee$, and $m : X \to \Omega$, define

$$(m \oplus q)y = \bigvee_{x \in X} q(x, y)m(x)$$

$$(q \cdot q')(x, z) = \bigvee_{y \in X} q'(y, z) \circ q(x, y)$$

gives a quantale and a quantale module in case of discrete set $X$

This extends to case of monotone relations when $X$ is $(X, \leqslant)$

$$X^{\mathsf{op}} \times X \to [\Omega, \Omega]_{\vee}$$

Now $\Omega^X$ means monotone functions, which makes sense for graphs

# $\mathcal{Q}$-category

Assume a **quantale** $\mathcal{Q} = (Q, \leqslant, \otimes, \mathbf{1})$ where $\otimes$ is commutative and has an identity $\mathbf{1}$ which is also the top element of the lattice.

(Think: truth values rather than relations)

Definition

*A* $\mathcal{Q}$-**category** *consists of a set* $X$ *and a function* $\mathcal{X} : X \times X \to Q$ *such that for all* $x, y, z \in X$*,*

1. $\mathbf{1} \leqslant \mathcal{X}(x, x)$*, and*
2. $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \leqslant \mathcal{X}(x, z)$*.*

Every $\mathcal{Q}$-category has a **dual** defined by $\mathcal{X}^{\mathsf{op}}(x, y) = \mathcal{X}(y, x)$.

### Definition

*Given $\mathcal{Q}$-categories $\mathcal{X}, \mathcal{Y}$,*

*a $\mathcal{Q}$-**functor** $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is*

*a function $F : X \to Y$ which for all $x_1, x_2 \in X$ satisfies*

$$\mathcal{X}(x_1, x_2) \leqslant \mathcal{Y}(Fx_1, Fx_2).$$

# $\mathcal{Q}$ as $\mathcal{Q}$-category

For each $q \in Q$, the map $p \mapsto p \otimes q$ has a right adjoint $r \mapsto q \Rightarrow r$ so that

$$p \leqslant q \Rightarrow r \text{ iff } p \otimes q \leqslant r.$$

The binary operation $\Rightarrow : Q \times Q \to Q$ makes the quantale $\mathcal{Q}$ itself a $\mathcal{Q}$-category with $\mathcal{Q}(p, q) = p \Rightarrow q$.

# $\mathcal{X} \otimes \mathcal{Y}$ for $\mathcal{Q}$-categories

Given $\mathcal{Q}$-categories $\mathcal{X}, \mathcal{Y}$, the $\mathcal{Q}$-category $\mathcal{X} \otimes \mathcal{Y}$ consists of the set $X \times Y$ equipped with the function $\mathcal{X} \otimes \mathcal{Y} : (X \times Y) \times (X \times Y) \to X \times Y$ where

$$(\mathcal{X} \otimes \mathcal{Y})((x_1, y_1), (x_2, y_2)) = \mathcal{X}(x_1, x_2) \otimes \mathcal{Y}(y_1, y_2).$$

### Definition

*A $\mathcal{Q}$-distributor is a $\mathcal{Q}$-functor, $\mathcal{R}$ of the form*

$$\mathcal{R} : \mathcal{X}^{op} \otimes \mathcal{Y} \to \mathcal{Q}.$$

*The notation $\mathcal{R} : \mathcal{X} \looparrowright \mathcal{Y}$ is used to indicate a $\mathcal{Q}$-distributor.*

Given $\mathcal{Q}$-distributors $\mathcal{R} : \mathcal{X} \looparrowright \mathcal{Y}$ and $\mathcal{S} : \mathcal{Y} \looparrowright \mathcal{Z}$ their composite is defined as

$$(R\,;S)(x,z) = \bigvee_{y \in Y} \left(\mathcal{R}(x,y) \otimes \mathcal{S}(y,z)\right).$$

but we actually need $\mathcal{Q}$-functors $\mathcal{X}^{\mathsf{op}} \otimes \mathcal{X} \to [\mathcal{Q}, \mathcal{Q}]_{\vee}$

...