Normal Forms for Elements of the *-continuous Kleene Algebras $K \otimes_{\mathcal{R}} C_2'$

Mark Hopkins and Hans Leiß

leiss@cis.uni-muenchen.de
2017 retired from: Universität München,
Centrum für Informations- und Sprachverarbeitung

RAMiCS 2023, April 3–6, Augsburg, Germany

Content

- Algebraization of formal language theory: categories $\mathbb{D}\mathcal{A}$ of $\mathcal{A}$-dioids having quotients (coequalizers) and tensor products.
- The polycyclic $\mathcal{R}$-dioid $C_2' = \mathcal{R}\Delta_2^*/\rho_2$ of 2 bracket pairs
- Automata $\langle S, A, F\rangle$ over $K \otimes_{\mathcal{R}} C_2'$ with $SA^*F \in K \otimes_{\mathcal{R}} C_2'$
- First Normal form for SA^*F with $A = (U + X + V)$:

$$(U + X + V)^* = (NV)^* N (UN)^* \quad \text{for } N = \mu y.(UyV + X)^*$$

- Reduced normal form for $SA^*F \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$: SNF
- Regular combinations of normal forms

Algebraization of Formal Language Theory

- $\mathbb{M}$ the category of monoids $(M, \cdot, 1)$ and homomorphisms,
- $\mathbb{D}$ the category of dioids = idempotent semirings $(D, +, \cdot, 0, 1)$ and semiring homomorphisms.

A monadic opertor $\mathcal{A}$ (Hopkins 2008) is a functor $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ that satisfies, for all monoids M, N and homomorphisms $f : M \to N$,

- A_0 $\mathcal{A}M$ is a set of subsets of M,
- A_1 $\mathcal{A}M$ contains each finite subset of M (hence $\emptyset, \{1\}$)
- A_2 $\mathcal{A}M$ is closed under product (hence a monoid),
- A_3 $\mathcal{A}M$ is closed under union of sets from $\mathcal{A}\,\mathcal{A}M$ (hence a dioid),
- A_4 $\mathcal{A}f := \lambda U\ \{f(m) \mid m \in U\} : \mathcal{A}M \to \mathcal{A}N$ is a homomorphism.

Theorem (Hopkins 2008)

$\mathcal{F}$(finite), $\mathcal{R}$(regular), $\mathcal{C}$(context-free), $\mathcal{T}$(r.e.), $\mathcal{P}$(all sets) are monadic operators. [$\mathcal{S}$(context-sensitive) does not satisfy A_4]

The category $\mathbb{D}\mathcal{A} \subseteq \mathbb{D}$ of $\mathcal{A}$-dioids

An $\mathcal{A}$-dioid is a partially ordered monoid $M = (M, \cdot, 1, \leq)$ which is

- $\mathcal{A}$-complete: every $U \in \mathcal{A}M$ has a supremum $\sum U \in M$, and
- $\mathcal{A}$-distributive: for all $U, V \in \mathcal{A}M$, $\sum(UV) = (\sum U)(\sum V)$.

$$\text{equivalently: for all } a, b \in M, U \in \mathcal{A}M : a(\sum U)b = \sum aUb.$$

Write $\mathcal{A}$-dioids as dioids $D = (M, +, \cdot, 0, 1)$, with $0, +$ given by $\sum$, and $\mathcal{A}D$ for $\mathcal{A}(M, \cdot, 1)$.

For $\mathcal{A}$-dioids D, D', an $\mathcal{A}$-morphism $f : D \to D'$ is a monotone homomorphism which is $\mathcal{A}$-continuous, i.e. which satisfies

$$\text{for all } U \in \mathcal{A}D: \quad f(\sum U) = \sum{}'(\mathcal{A}f)(U).$$

Let $\mathbb{D}\mathcal{A}$ be the category of $\mathcal{A}$-dioids and $\mathcal{A}$-morphisms. ($\mathbb{D}\mathcal{F} = \mathbb{D}$.)

$\mathbb{D}\mathcal{A}$ has quotients $D/_\equiv$ by $\mathcal{A}$-congruences, and tensor products.

Theorem (MH,HL 2018)

In $\mathbb{D}\mathcal{A}$, a tensor product $\top_1 : D_1 \to D_1 \otimes_{\mathcal{A}} D_2 \leftarrow D_2 : \top_2$ of $\mathcal{A}$-dioids D_1, D_2 exists, and it consists of

- $D_1 \otimes_{\mathcal{A}} D_2 := \mathcal{A}(M_1 \times M_2)/_\equiv$, *where M_i is the multiplicative monoid of D_i and $\equiv$ is the least $\mathcal{A}$-congruence s.th.*

$$\{(\sum A, \sum B)\} \equiv A \times B, \quad \text{for all } A \in \mathcal{A}M_1, B \in \mathcal{A}M_2,$$

- *commuting morphisms $\top_1, \top_2$ given by* $\top_1(a) := \{(a,1)\}/_\equiv,\ \top_2(b) = \{(1,b)\}/_\equiv$, *for $a \in D_1, b \in D_2$.*

The induced map of $f : D_1 \to D \leftarrow D_2 : g$ is

$$h_{f,g}(U/_\equiv) := \sum \{f(a)g(b) \mid (a,b) \in U\}, \quad U \in \mathcal{A}(M_1 \times M_2).$$

Write $a \otimes b := \{(a,b)\}/_\equiv,\ [U] := U/_\equiv = \sum\{a \otimes b \mid (a,b) \in U\}$.

Theorem (Hopkins 2008, L./Hopkins 2018)

- *$\mathbb{D}\mathcal{R}$ is the category of *-continuous Kleene algebras.*
- *$\mathbb{D}\mathcal{C}$ is the category of μ-continuous Chomsky algebras.*

Kleene-Algebra (Kozen 1990): right- and left-linearly closed dioid

$x \geq ax + b$ and $x \geq xa + b$ have least solutions a^*b resp. ba^*, for all values a, b.

*-continuity: $a \cdot c^* \cdot b = \sum\{a \cdot c^n \cdot b \mid n \in \mathbb{N}\}$, for all $a, b, c \in M$.

Chomsky-Algebra (Grathwohl e.a. 2015): algebraically closed dioid

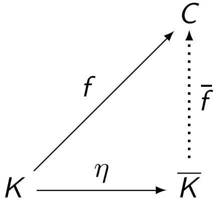
every polynomial system $x_1 \geq p_1(\bar{x}, \bar{y}), \ldots, x_n \geq p_n(\bar{x}, \bar{y})$ has a least solution in $\bar{x} = x_1, \ldots x_n$, for each value of $\bar{y}$.

μ-continuity: $a \cdot \mu x p \cdot b = \sum\{a \cdot p^n(0) \cdot b \mid n \in \mathbb{N}\}$, all $p \in M[x]$.

There is an adjunction between $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$,

$$Q^{\mathcal{C}}_{\mathcal{R}} : \mathbb{D}\mathcal{R} \rightleftarrows \mathbb{D}\mathcal{C} : Q^{\mathcal{R}}_{\mathcal{C}},$$

where $Q^{\mathcal{C}}_{\mathcal{R}}(K)$ is the $\mathcal{C}$-completion of K, i.e. a $\mathcal{C}$-dioid $\overline{K}$ with an $\mathcal{R}$-morphism $\eta : K \to \overline{K}$ such that any $\mathcal{R}$-morphism $f : K \to C$ to a $\mathcal{C}$-dioid C extends uniquely to a $\mathcal{C}$-morphism $\bar{f} : \overline{K} \to C$, i.e. $f = \bar{f} \circ \eta$:



Prop. For monoids M, the $\mathcal{C}$-completion of $\mathcal{R}M$ is $\mathcal{C}M$, with

$$\bar{f}(L) = \sum \{ f(\{m\}) \mid m \in L \}, \quad \text{for } L \in \mathcal{C}M.$$

C_2' and the Representation of the $\mathcal{C}$-Completion of $K \in \mathbb{DR}$

Let $\Delta_n = P_n \dot\cup Q_n$, for $P_n = \{p_0, \ldots, p_{n-1}\}, Q_n = \{q_0, \ldots, q_{n-1}\}$, and $(\Delta_n^*)_0$ the extension of Δ_n^* by an annihilating element 0.

The polycyclic $\mathcal{R}$-dioid C_n' is $\mathcal{R}\Delta_n^*/\rho_n$, with $\mathcal{R}$-congruence ρ_n from

$$\{p_i\}\{q_i\} = \{1\}, \qquad \{p_i\}\{q_j\} = \emptyset, \quad (i \neq j).$$

Let $nf : \Delta_n^* \to Q_n^* P_n^* \cup \{0\}$ normalize strings via $p_i q_j \to \delta_{i,j}$. Then $A \in \mathcal{R}\Delta_n^*$, $A/\rho_n \in C_n'$ is represented by $\{nf(w) \mid w \in A\} \setminus \{0\}$,

The pure Dyck-language $D \in \mathcal{C}\Delta_n^*$ is $\{w \in \Delta_n^* \mid nf(w) = 1\}$.

Prop. There is an embedding $\mathcal{R}$-morphism $\overline{\cdot} : C_n' \to C_2'$ based on coding the p_i, q_i of Δ_n by the two pairs b, d and p, q of Δ_2 via

$$\overline{p_i} := bp^{i+1} \in P_2^* p, \qquad \overline{q_i} := q^{i+1} d \in q Q_2^*.$$

For $\mathcal{A}$-dioids D, C, the centralizer of C in $D \otimes_{\mathcal{A}} C$ is

$$Z_C(D \otimes_{\mathcal{A}} C) := \{\varphi \in D \otimes_{\mathcal{A}} C \mid \varphi(1 \otimes c) = (1 \otimes c)\varphi \text{ for all } c \in C\}.$$

Lemma

$$Z_{C_2'}(K \otimes_{\mathcal{R}} C_2') \simeq \{[R] \mid R \in \mathcal{R}(K \times C_2'), R \subseteq K \times \{0, 1\}\}.$$

Theorem (Algebraic representation of $\mathcal{C}$-completion of $\mathcal{R}X^*$)

$$\mathcal{C}X^* \simeq Z_{C_2'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_2') \quad \textit{via } L \mapsto \sum\{\{w\} \otimes 1 \mid w \in L\}$$

Theorem (Algebraic representation of the $\mathcal{C}$-completion of K)

- $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ *is a* $\mathcal{C}$*-dioid, i.e.* μ*-continuous Chomsky algebra,*
- $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ *is the* $\mathcal{C}$*-completion of* **-cont. Kleene algebra* K*.*

Use $\Delta_2 = \{\langle 0|, |0\rangle, \langle 1|, |1\rangle\}$. An element of $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ is the value of some $r \in RegExp(X \cup \Delta_2)$ in the generators X and Δ_2 of C_2'.

Let $x \in X$ stand for its image $\{x\} \otimes 1 \in \mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ and $t \in \Delta_2^*$ for its image $1 \otimes \{t\}/\rho_2$. We get for $L = \{a^n cb^n \mid n \in \mathbb{N}\} \in \mathcal{C}X^*$:

$$\begin{aligned}
\langle 0|(a\langle 1|)^* c(|1\rangle b)^*|0\rangle &= \sum_{n,m\in\mathbb{N}} \langle 0|(a\langle 1|)^n c(|1\rangle b)^m|0\rangle \quad (^*\text{-continuity}) \\
&= \sum_{n,m\in\mathbb{N}} a^n cb^m \underbrace{\langle 0|\langle 1|^n|1\rangle^m|0\rangle}_{\delta_{n,m}} \quad (x, t \text{ commute}) \\
&= \sum_{n\in\mathbb{N}} a^n cb^n \quad = \widehat{L} = [R] \quad \text{for}
\end{aligned}$$

$$R = \{(\{a^n cb^m\}, \{\langle 0|\langle 1|^n|1\rangle^m|0\rangle\}/\rho_2) \mid n, m \in \mathbb{N}\} \in \mathcal{R}(\mathcal{R}X^* \times C_2').$$

Automata $\langle S, A, F \rangle$ over $K \otimes_{\mathcal{R}} C_2'$

A finite automaton $\langle S, A, F \rangle$ with n states over a Kleene algebra K consists of a matrix $A \in Mat_{n,n}(K)$ and vectors $S \in Mat_{1,n}(\mathbb{B})$ and $F \in Mat_{n,1}(\mathbb{B})$ coding the initial and final states $i < n$.

$A_{i,j}$ represents the 1-step transitions from state i to state j, $A^*_{i,j}$ the "set" of paths of finite length from i to j, and $\langle S, A, F \rangle$ represents

$$S \cdot A^* \cdot F \ \in K.$$

The iteration M^* of $M \in Mat_{n,n}(K)$ is defined by induction on n as

$$M^* = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} F^* & F^*BD^* \\ D^*CF^* & D^*CF^*BD^* + D^* \end{pmatrix},$$

where $F = A + BD^*C$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A and D square.

Theorem (Kozen 1991, chapter 7.1)

If K is a *-continuous Kleene algebra, so is $Mat_{n,n}(K)$, for $n \geq 1$.

By Kleene's theorem,

$$\mathcal{R}X^* = \{SA^*F \mid \langle S, A, F\rangle \text{ an automaton with entries in } \mathcal{F}X^*\}.$$

Let K be an $\mathcal{R}$-dioid, $\Delta_2 = P_2 \dot{\cup} Q_2$ and $P_2 = \{b, p\}, Q_2 = \{d, q\}$.

For $a \in K$ and $t \in \Delta_2$, we write a and t also for their images $a \otimes 1$ and $1 \otimes \{t\}/\rho_2$ in $K \otimes_{\mathcal{R}} C_2'$.

Theorem (Representation of $\varphi \in K \otimes_{\mathcal{R}} C_2'$ by an automaton)

For each $\varphi \in K \otimes_{\mathcal{R}} C_2'$ there is an automaton $\langle S, U + X + V, F\rangle$ with n states, $U \in \{0, b, p\}^{n\times n}$, $X \in K^{n\times n}$, $V \in \{0, d, q\}^{n\times n}$ s.th.

$$\varphi = S(U + X + V)^*F.$$

Proof: by induction on the regular $R \in \mathcal{R}(K \times C_2')$ s.th. $\varphi = [R]$.

Normal Forms of Automata over $K \otimes_{\mathcal{R}} C_2'$

There are several ways to define Dyck's language $D \subseteq \{u, x, v\}^*$ with one "bracket" pair u, v in arbitrary Kleene algebras:

Prop. Let K be a Kleene algebra and $u, x, v \in K$.

(i) If $y \geq 1 + x + uyv + yy$ has a least solution D, then D is the least solution of $y \geq (x + uyv)^*$.

(ii) If $y \geq (x + uyv)^*$ has a least solution N, then N is the least solution of $y \geq 1 + x + uyv + yy$.

Notice: $\{u, x, v\}^* = (Dv)^* D(uD)^*$ for Dyck's $D \subseteq \{u, x, v\}^*$.

Theorem

Let K be a Kleene algebra and $u, x, v \in K$. If $y \geq (x + uyv)^*$ has a least solution N in K, then $(u + x + v)^* = (Nv)^* N(uN)^*$.

When multiplying b, d, p, q with $n \times n$-matrices, we identify them with corresponding diagonal matrices.

Lemma

Let K be an $\mathcal{R}$-dioid, $n \in \mathbb{N}$, $A = U + X + V$ with $U \in \{0, b, p\}^{n\times n}$, $V \in \{0, d, q\}^{n\times n}$ and $X \in K^{n\times n}$. In $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2')$,

$$y \geq (UyV + X)^* \tag{1}$$

has a least solution, namely $N := b(Up + X + qV)^* d$, and

$$N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}.$$

Proof: Let $D \subseteq \{U, X, V\}^*$ be Dyck's language with brackets U,V.
1. $N_m := b(Up + X + qV)^m d = \sum(\{U, X, V\}^m \cap D) \in K^{n\times n}$.
2. By *-continuity, $N = \sum D$ and $cN = Nc$ for $c \in C_2'$.
3. Show that N solves (1), since $N = \sum D$. □

Theorem (First Normal Form)

Let K be an $\mathcal{R}$-dioid. For $\varphi \in K \otimes_{\mathcal{R}} C_2'$ there are $n \in \mathbb{N}, S \in \mathbb{B}^{1\times n}$, $F \in \mathbb{B}^{n\times 1}$, $U \in \{0, b, p\}^{n\times n}$, $V \in \{0, d, q\}^{n\times n}$, $X \in K^{n\times n}$ s.th.

$$\varphi = S(NV)^* N(UN)^* F,$$

where $N \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}$ is $\mu y.(UyV + X)^*$.

For $n = 1$, N commutes with U, V, so $(NV)^* N(UN)^* = V^* NU^*$.

Proof.

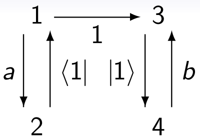
There is an automaton $\langle S, A, F\rangle$ for φ with $A = U + X + V$ as above and a least solution N of $y \geq (UyV + X)^*$ such that

$$A^* = (U + X + V)^* = (NV)^* N(UN)^*.$$

Hence $\varphi = SA^*F = S(NV)^*N(UN)^*F$. □

Example: Let $a, b \in K$, $\Delta_2 = \{\langle 0|, \langle 1|, |0\rangle|1\rangle\}$, and

$$A = (U + X + V) = \begin{pmatrix} 0 & a & 1 & 0 \\ \langle 1| & 0 & 0 & 0 \\ 0 & 0 & 0 & |1\rangle \\ 0 & 0 & b & 0 \end{pmatrix}$$



With $\bar{a} = a\langle 1|$, $\bar{b} = |1\rangle b$,

$$A^* = \begin{pmatrix} \bar{a}^* & \bar{a}^* a & \bar{a}^* \bar{b}^* & \bar{a}^* \bar{b}^* |1\rangle \\ \langle 1|\bar{a}^* & 1 + \langle 1|\bar{a}^* a & \langle 1|\bar{a}^* \bar{b}^* & \langle 1|\bar{a}^* \bar{b}^* |1\rangle \\ 0 & 0 & \bar{b}^* & \bar{b}^* |1\rangle \\ 0 & 0 & b\bar{b}^* & 1 + b\bar{b}^* |1\rangle \end{pmatrix}$$

With $N = \langle 0|(U\langle 1| + X + |1\rangle V)^* |0\rangle$ and $\widehat{L} = \langle 0|(a\langle 1|^2)^* (|1\rangle^2 b)^* |0\rangle = \sum\{a^n b^n \mid n \in \mathbb{N}\}$,

$(NV)^* N (UN)^* =$

$$\begin{pmatrix} 0 & 0 & 0 & \widehat{L}|1\rangle \\ 0 & 0 & 0 & \widehat{L}b|1\rangle \\ 0 & 0 & 0 & |1\rangle \\ 0 & 0 & 0 & b|1\rangle \end{pmatrix}^* \begin{pmatrix} 1 & a & \widehat{L} & a\widehat{L} \\ 0 & 1 & \widehat{L}b & \widehat{L} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \langle 1| & \langle 1|a & \langle 1|\widehat{L} & \langle 1|a\widehat{L} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^*$$

Corollary (Reduced Normal Form)

Let Δ_m have the bracket pairs $\langle i|, |i\rangle$ for $i = 0, \ldots, m-1$. Suppose $\varphi \in K \otimes_{\mathcal{R}} C'_m$ is represented by $\langle S, A, F\rangle$ not using $\langle 0|, |0\rangle$, i.e. $A = U + X + V$ with $U \in \{0, \langle 1|, \ldots \langle m-1|\}^{n\times n}$, $X \in K^{n\times n}$, $V \in \{0, |1\rangle, \ldots, |m-1\rangle\}^{n\times n}$. If $N = \mu y.(UyV + X)^*$, then

$$\langle 0|\varphi|0\rangle = SNF \in Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m).$$

If moreover $\varphi \in Z_{C'_m}(K \otimes_{\mathcal{R}} C'_m)$, then $\varphi = \langle 0|\varphi|0\rangle = SNF$.

Proof.

By the assumption on U and V, $\langle 0|V = 0 = U|0\rangle$. As N commutes with $\langle 0|$ and $|0\rangle$, we get $\langle 0|(NV)^* = \langle 0|$ and $(UN)^*|0\rangle = |0\rangle$. Hence

$$\begin{aligned} \langle 0|A^*|0\rangle &= \langle 0|(NV)^* N (UN)^*|0\rangle = \langle 0|N|0\rangle = N, \\ \langle 0|\varphi|0\rangle &= \langle 0|SA^*F|0\rangle = S\langle 0|A^*|0\rangle F = SNF. \end{aligned}$$

[Using $\bar{\cdot} : C'_m \to C'_2$, we may admit $\langle 0|$ in U and $|0\rangle$ in V.] $\square$

In the monoid case, we have: $\varphi \in Z_{P'_m}(X^* \times P'_m) = X^* \cup \{0\}$ iff $\varphi = \langle 0|w|0\rangle$ for some $w \in (X \cup \{0\} \cup \Delta_m)^*$ without $\langle 0|, |0\rangle$.

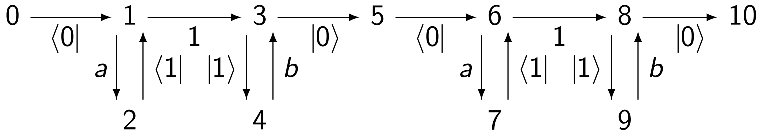
Theorem

Suppose $\varphi \in \mathcal{R}X^* \otimes_{\mathcal{R}} C'_m$, $m > 2$. Then $\varphi \in Z_{C'_m}(\mathcal{R}X^* \otimes_{\mathcal{R}} C'_m)$ iff $\varphi = \langle 0|r|0\rangle$ for some $r \in RegExp(X \dot\cup \Delta_m)$ not containing $\langle 0|, |0\rangle$.

A Second Normal Form can be given for automata with transitions by $|0\rangle\langle 0|$ in addition to those by elements of K and $\Delta_m \setminus \{\langle 0|, |0\rangle\}$. If $W \in \mathbb{B}^{n\times n}$, $\varphi = S(A + |0\rangle\langle 0|W)^* F$, then $\langle 0|\varphi|0\rangle = SN(WN)^* F$.

So we can combine representations $\langle 0|r_i|0\rangle = \sum L_i$ of $L_i \in \mathcal{C}X^*$ in $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_2$ to a representation of $L_1 L_2$,

$$\langle 0|r_1|0\rangle\langle 0|r_2|0\rangle = (\sum L_1)(\sum L_2) = \sum (L_1 L_2).$$



Calculus for context-free expressions for $\mathcal{C}X^*$

The normal forms of $a \in K$, $c \in C_2'$, $(\varphi_1 + \varphi_2)$, $(\varphi_1 \cdot \varphi_2)$ and φ_1^* can be obtained from those of their components.

Suppose $\varphi_i \in K \otimes_{\mathcal{R}} C_2'$ is $S_i A_i^* F_i$ with $A_i = U_i + X_i + V_i$ and $N_i = \mu y.(U_i y V_i + X_i)^*$, so that $A^* = (N_i V_i)^* N_i (U_i N_i)^*$.

Define S, F, U, X, V, N for $(\varphi_1 + \varphi_2)$, $(\varphi_1 \cdot \varphi_2)$, and φ_1^+ by

$$(S_1 \ S_2), \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

$$(S_1 \ 0), \begin{pmatrix} 0 \\ F_2 \end{pmatrix}, \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \begin{pmatrix} X_1 & F_1 S_2 \\ 0 & X_2 \end{pmatrix}, \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \begin{pmatrix} N_1 & N' \\ 0 & N_2 \end{pmatrix}$$

with $N' = \mu y.(N_1 U_1 y V_2 N_2 + N_1 F_1 S_2 N_2) \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')^{n_1 \times n_2}$.

$$S_1, \quad F_1, \quad U_1, \quad X_1 + F_1 S_1, \quad V_1, \quad N'$$

with $N' = \mu y.(U_1 y V_1 + N_1 + F_1 S_1)^* \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')^{n_1 \times n_1}$.

The two N' above exist, since the corresponding matrix inequations

$$\begin{aligned} y &\geq (N_1 U_1 y V_2 N_2 + N_1 F_1 S_2 N_2) \quad \text{resp.} \\ y &\geq (U_1 y V_1 + N_1 + F_1 S_1)^* \end{aligned}$$

amount to[1] polynomial systems

$$y_1 \geq p_1(\bar{y}, \bar{b}), \ldots, y_m \geq p_m(\bar{y}, \bar{b})$$

with parameters $\bar{b}$ from the $\mathcal{C}$-dioid $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$. In general,

Theorem (CSL 2016)

For all $\mathcal{C}$-dioids D and $n \in \mathbb{N}$, $Mat_{n,n}(D)$ is a $\mathcal{C}$-dioid.

Hence, for $\mathcal{R}$-dioid K, $Mat_{n,n}(Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))$ is a $\mathcal{C}$-dioid.

[1] entries of $U_i y V_j$ are polynomials in 0, 1 and $y_{k,l}$'s, since $p_i y_{k,l} q_j = y_{k,l} \delta_{i,j}$

Conclusion

- The fixpoint completion of $\mathcal{R}$-dioid K can be represented inside the tensor product $K \otimes_{\mathcal{R}} C_2'$ by the centralizer of C_2',

$$Z_{C_2'}(K \otimes_{\mathcal{R}} C_2') \subseteq K \otimes_{\mathcal{R}} C_2'.$$

- Elements φ of $K \otimes_{\mathcal{R}} C_2'$ can be represented as $\varphi = SA^*F$ for automata $\langle S, A, F\rangle$ with $A = U + X + V$, $U \in \{0, b, p\}^{n\times n}$, $V \in \{0, d, q\}^{n\times n}$, $X \in K^{n\times n}$ for some n, where
  - $A^* = (NV)^*N(UN)^*$ gives a normal form for $\varphi = SA^*F$
  - if $\varphi \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$, then $\varphi = SA^*F = SNF$, [almost]

  for $N = \mu y.(UyV + X)^* \in (Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))^{n\times n}$
- Normal forms can be built inductively from K, C_2' via $+$, $\cdot$, *.

Open Problems

- $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ as an algebra for recognizers/parsers for $\mathcal{C}X^*$
- $\mathcal{R}X^* \otimes_{\mathcal{R}} \mathcal{R}Y^* \otimes_{\mathcal{R}} C_2'$ as an algebra of translations $\subseteq X^* \times Y^*$
- Construct $Q_{\mathcal{C}}^{\mathcal{T}} : \mathbb{D}\mathcal{C} \to \mathbb{D}\mathcal{T}$, a category of cont. Turing algebras

M. Hopkins. The algebraic approach I: The algebraization of the Chomsky hierarchy. II: Dioids, quantales and monads.
In Proc. *Relational Methods in Computer Science/Applications of Kleene Algebra*, LNCS 4988, pp. 155–190. Springer 2008.

M. Hopkins and H. Leiß. Coequalizers and tensor products for continuous idempotent semirings.
In Proc. *17th Int. Conf. on Relational and Algebraic Methods in Computer Science*, LNCS 11194, 37–52. Springer 2018.

H. Leiß. An algebraic representation of the fixed-point closure of *-continuous Kleene algebras.
Mathematical Structures in Computer Science, vol. 32.6, 2022
https://www.cis.uni-muenchen.de/~leiss/MSCS-2020-072.R2.rev279.pdf (submitted version)

H. Leiß and M. Hopkins. C-dioids and μ-continuous Chomsky algebras.
In Proc. *17th Int. Conf. on Relational and Algebraic Methods in Computer Science, RAMiCS 2018*, 21–36. Springer 2018.