An Algebraic Representation of the Fixed-Point Closure of *-Continuous Kleene Algebras

A Categorical Chomsky-Schützenberger Theorem

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Content

- The Chomsky-Schützenberger-Theorem: how to obtain $\mathcal{C}X^*$ from $\mathcal{R}(X \dot{\cup} \Delta)^*$ and Dyck's language $D_X \in \mathcal{C}(X \dot{\cup} \Delta)^*$
- Subcategories $\mathbb{D}\mathcal{A}$ of the category $\mathbb{D}$ of idempotent semirings
  - $\mathbb{D}\mathcal{R} =$ *-continuous Kleene algebras
  - $\mathbb{D}\mathcal{C} = \mu$-continuous Chomsky algebras

  There is an adjunction $Q^{\mathcal{C}}_{\mathcal{R}} : \mathbb{D}\mathcal{R} \rightleftarrows \mathbb{D}\mathcal{C} : Q^{\mathcal{R}}_{\mathcal{C}}$ where
  - $Q^{\mathcal{C}}_{\mathcal{R}}$ gives the $\mathcal{C}$-completion or "fixed-point closure"
  - $Q^{\mathcal{R}}_{\mathcal{C}}$ is the forgetful functor (aka restriction of μ to *)
- Algebraic representation: $Q^{\mathcal{C}}_{\mathcal{R}}(K) = Z_{\mathcal{R}\Delta^*/\rho}(K \otimes_{\mathcal{R}} \mathcal{R}\Delta^*/\rho)$
- For $K = \mathcal{R}X^*$: $\mathit{RegExp}(X \dot{\cup} \Delta)$ name $\mathcal{C}X^* = Q^{\mathcal{C}}_{\mathcal{R}}(\mathcal{R}X^*)$

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The classical CST for free monoids

$$\begin{aligned} X^* &= (X^*, \cdot, 1) \text{ the free monoid generated by the fin.set } X \\ M[\Delta] &= \text{the free extension of the monoid } M \text{ by the set } \Delta \\ &= \text{all interleaved sequences of elements of } M \text{ and } \Delta^* \end{aligned}$$

Theorem (Chomsky/Schützenberger 1963)

Let X be a finite set and

- $\Delta_2 = \{b, d, p, q\}$ *a set of two bracket pairs b, d and p, q,*
- $h_{X^*} : X^*[\Delta_2] \to X^*$ *the bracket-erasing homomorphism,*
- $D_X \in \mathcal{C}(X^*[\Delta_2])$ *Dyck's language, the least $S \subseteq X^*[\Delta_2]$ s.th.*

$$S \geq 1 + X + bSd + pSq + SS.$$

Then: $\quad \mathcal{C}X^* = \{h_{X^*}(R \cap D_X) \mid R \in \mathcal{R}(X^*[\Delta_2])\}.$

Our goal: an *algebraic* construction of $\mathcal{C}X^*$ from $\mathcal{R}X^*$ itself.

Categories $\mathbb{D}\mathcal{A}$ of Dioids

- $\mathbb{M}$ the category of monoids $(M, \cdot, 1)$ and homomorphisms,
- $\mathbb{D}$ the category of dioids = idempotent semirings $(D, +, \cdot, 0, 1)$ and semiring homomorphisms.

A monadic opertor $\mathcal{A}$ (Hopkins 2008) is a functor $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ such that for all monoids M, N and homomorphisms $f : M \to N$

- A_0 $\mathcal{A}M$ is a set of subsets of M,
- A_1 $\mathcal{A}M$ contains each finite subset of M (hence $\emptyset, \{1\}$),
- A_2 $\mathcal{A}M$ is closed under elem.wise product (hence a monoid),
- A_3 $\mathcal{A}M$ is closed under union of sets from $\mathcal{A}(\mathcal{A}M)$ (hence a dioid),
- A_4 $\mathcal{A}f := \lambda U \, \{f(m) \mid m \in U\} : \mathcal{A}M \to \mathcal{A}N$ is a homomorphism.

Theorem (Hopkins 2008)

$\mathcal{F}$ (finite), $\mathcal{R}$ (regular), $\mathcal{C}$ (context-free), $\mathcal{T}$ (r.e.), $\mathcal{P}$ (all sets) are monadic operators. [$\mathcal{S}$ (context-sensitive) does not satisfy A_4]

An $\mathcal{A}$-dioid is a partially ordered monoid $M = (M, \cdot, 1, \leq)$ which is

- $\mathcal{A}$-complete: every $U \in \mathcal{A}M$ has a supremum $\sum U \in M$, and
- $\mathcal{A}$-distributive: for all $U, V \in \mathcal{A}M$, $\sum(UV) = (\sum U)(\sum V)$.

equivalently: for all $a, b \in M, U \in \mathcal{A}M : a(\sum U)b = \sum aUb.$

Prop. $\mathcal{A}M$, $m \mapsto \{m\}$, is the $\mathcal{A}$-dioid completion of the monoid M.

Notation: $\mathcal{A}$-dioids $D = (M, +, \cdot, 0, 1)$ as dioids, with $0, +$ via $\sum$,
$\mathcal{A}D$ for $\mathcal{A}(M, \cdot, 1)$.

For $\mathcal{A}$-dioids D, D', an $\mathcal{A}$-morphism $f : D \to D'$ is a monotone homomorphism which is $\mathcal{A}$-continuous, i.e.

$$f(\sum U) = \sum{}'(\mathcal{A}f)(U) \quad \text{for all } U \in \mathcal{A}D..$$

Let $\mathbb{D}\mathcal{A}$ be the category of $\mathcal{A}$-dioids and $\mathcal{A}$-morphisms. ($\mathbb{D}\mathcal{F} = \mathbb{D}$.)

Theorem (Hopkins 2008, L./Hopkins 2018)

- $\mathbb{D}\mathcal{R}$ *is the category of* **-continuous Kleene algebras.*
- $\mathbb{D}\mathcal{C}$ *is the category of* μ*-continuous Chomsky algebras.*

Kleene-Algebra (Kozen 1990): right-/left-linearly closed dioid D

$$\begin{array}{l} x \geq ax + b \\ x \geq xa + b \end{array} \quad \textit{has least solution} \quad \begin{array}{l} a^* b \\ ba^* \end{array}, \textit{ for all } a, b \in D.$$

*-continuity: $a \cdot c^* \cdot b = \sum \{a \cdot c^n \cdot b \mid n \in \mathbb{N}\}$, for all $a, b, c \in D$.

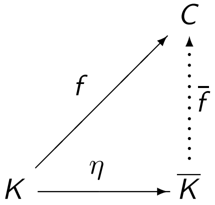
Chomsky-Algebra (Grathwohl e.a. 2015): algebraically closed dioid

every polynomial system $x_1 \geq p_1(\bar{x}, \bar{y}), \ldots, x_n \geq p_n(\bar{x}, \bar{y})$
has a least solution in $\bar{x} = x_1 \ldots x_n$, *for all values of* $\bar{y}$ *in* D.

μ-continuity: $a \cdot \mu x p \cdot b = \sum \{a \cdot p^n(0) \cdot b \mid n \in \mathbb{N}\}$, all $p \in D[x]$.

"Fixed-point-closure" $= \mathcal{C}$-completion

A $\mathcal{C}$-completion of $\mathcal{R}$-dioid K is a $\mathcal{C}$-dioid $\overline{K}$ with an $\mathcal{R}$-morphism $\eta : K \to \overline{K}$ such that any $\mathcal{R}$-morphism $f : K \to C$ to a $\mathcal{C}$-dioid C extends uniquely to a $\mathcal{C}$-morphism $\overline{f} : \overline{K} \to C$, i.e. $f = \overline{f} \circ \eta$:



Prop. For monoids M, the $\mathcal{C}$-completion of $\mathcal{R}M$ is $\mathcal{C}M$, with

$$\overline{f}(L) = \sum \{ f(\{m\}) \mid m \in L \}, \qquad \text{for } L \in \mathcal{C}M,$$

Theorem (Hopkins 2008)

The $\mathcal{C}$-completion is part of an adjunction $Q^{\mathcal{C}}_{\mathcal{R}} : \mathbb{D}\mathcal{R} \rightleftarrows \mathbb{D}\mathcal{C} : Q^{\mathcal{R}}_{\mathcal{C}}$.

The polycyclic monoid P'_n and $\mathcal{A}$-dioid $C'_{n,\mathcal{A}} = \mathcal{A}\Delta_n^*/\rho_n$

We are looking for an algebra in which $h_{X^*}(R \cap D_X)$ can be done.

Idea: Use alphabets $\Sigma = X \dot\cup \Delta_n$ of letters X and brackets Δ_n, and languages over Σ in which *letters commute with brackets*.

Let $\Delta_n = P_n \dot\cup Q_n$, for $P_n = \{p_0, \dots, p_{n-1}\}, Q_n = \{q_0, \dots, q_{n-1}\}$, and $(\Delta_n^*)_0$ the extension of Δ_n^* by an annihilating element 0.

The polycyclic monoid P'_n is the quotient monoid $(\Delta_n^*)_0/\rho_n$ where

$$\rho_n = \{p_i q_i = 1 \mid i < n\} \cup \{p_i q_j = 0 \mid i, j < n, i \neq j\}.$$

In P'_n each $w \in \Delta_n^*$ has a *normal form*

$$nf(w) \in \{0\} \cup Q_n^* P_n^*,$$

obtained by cancelling matching brackets $p_i q_i = 1$ (resp. $p_i q_j = 0$).

The normal form $nf(w)$ represents the element w/ρ_n of P'_n. Hence

$$P'_n \simeq (Q_n^* P_n^* \cup \{0\}, \cdot, 1) \quad \text{with } u \cdot v = nf(uv)$$

This extends to $X^*[\Delta_n]_0$ where letters commute with brackets, and nf commutes p_i to the right, q_j to the left, and applies $p_i q_j = \delta_{i,j}$:

$$\begin{aligned} X^* \times P'_n &:= X^*[\Delta_n]_0/(\rho_n \cup \{wt = tw \mid w \in X^*, t \in \Delta_n\}) \\ &\simeq (Q_n^* X^* P_n^* \cup \{0\}, \cdot, 1) \quad \text{with } u \cdot v = nf(uv) \end{aligned}$$

For new p_n, q_n: $R \subseteq X^*[\Delta_n] \Rightarrow nf(p_n R q_n) \setminus \{0\} = h_{X^*}(R \cap D_X)$.

The polycyclic $\mathcal{R}$-dioid C'_n is an "$\mathcal{R}$-quotient" of $\mathcal{R}\Delta_n^*$ resp. $\mathcal{R}P'_n$.

Then: lift $X^* \times P'_n$ to an "$\mathcal{R}$-tensor product" $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_n$ of $\mathcal{R}X^*$ and C'_n where $A \in \mathcal{R}X^*$ and (the quotient of) $B \in \mathcal{R}P'_n$ commute.

An $\mathcal{A}$-congruence on an $\mathcal{A}$-dioid D is a dioid-congruence ρ s.th. for all $U, V \in \mathcal{A}D$, if $(U/\rho)^{\downarrow} = (V/\rho)^{\downarrow}$, then $(\sum U)/\rho = (\sum V)/\rho$.

Prop. If D is an $\mathcal{A}$-dioid and ρ an $\mathcal{A}$-congruence on D, then D/ρ is an $\mathcal{A}$-dioid and the canonical map $d \mapsto d/\rho$ is an $\mathcal{A}$-morphism.

For any $E \subseteq D \times D$, there is a least $\mathcal{A}$-congruence $\rho \supseteq E$ on D.

The polycyclic $\mathcal{R}$-dioid C_n' is $\mathcal{R}\Delta_n^*/\rho_n$, with $\mathcal{R}$-congruence ρ_n by

$$\{p_i\}\{q_i\} = \{1\}, \qquad \{p_i\}\{q_j\} = \emptyset, \quad (i \neq j).$$

For $A \in \mathcal{R}\Delta_n^*$, $A/\rho_n \in C_n'$ is represented by $\{nf(w) \mid w \in A\} \setminus \{0\}$.

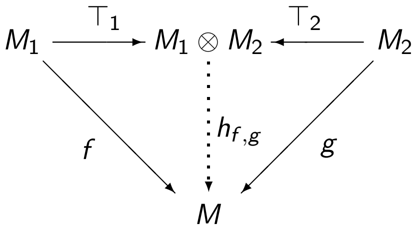
Prop. $C_n' \simeq \mathcal{R}P_n'/(\{0\} = \emptyset)$ where $P_n' \simeq (Q_n^* P_n^* \cup \{0\}, \cdot, 1)$.

The Tensor Product $D_1 \otimes_{\mathcal{A}} D_2$ of $\mathcal{A}$-Dioids

In a category $\mathbb{C}$ with reducts in $\mathbb{M}$, a tensor product of M_1, M_2, consists of an object $M_1 \otimes M_2$ with two commuting[1] morphisms

$$\top_1 : M_1 \to M_1 \otimes M_2 \leftarrow M_2 : \top_2,$$

such that any pair $f : M_1 \to M \leftarrow M_2 : g$ of commuting morphisms decompose with a unique induced morphism $h_{f,g}$ as shown:



[1] i.e. $\top_1(a)\top_2(b) = \top_2(b)\top_1(a)$ for all $a \in M_1, b \in M_2$

Theorem (MH,HL 2018)

The tensor product $D_1 \otimes_{\mathcal{A}} D_2$ of $\mathcal{A}$-dioids D_1, D_2 consists of

- $D_1 \otimes_{\mathcal{A}} D_2 = \mathcal{A}(M_1 \times M_2)/_{\equiv}$, *where M_i is the multiplicative monoid of D_i and $\equiv$ is the least $\mathcal{A}$-congruence s.th.*

$$\{(\sum A, \sum B)\} \equiv A \times B, \quad \textit{for all } A \in \mathcal{A}M_1, B \in \mathcal{A}M_2,$$

- *the commuting morphisms $\top_1 : D_1 \to D_1 \otimes_{\mathcal{A}} D_2 \leftarrow D_2 : \top_2$,*

$$a \mapsto \{(a,1)\}/_{\equiv} \textit{ and } b \mapsto \{(1,b)\}/_{\equiv}, \qquad \textit{for } a \in D_1, b \in D_2.$$

The induced morphism of $f : D_1 \to D \leftarrow D_2 : g$ is

$$h_{f,g}(U/_{\equiv}) = \sum\{f(a)g(b) \mid (a,b) \in U\}, \quad U \in \mathcal{A}(M_1 \times M_2).$$

Notation:

$$a \otimes b := T_1(a)T_2(b) = \{(a,b)\}/_{\equiv}$$

$$[U] := U/_{\equiv} = \sum\{a \otimes b \mid (a,b) \in U\}$$

Algebraic Representation of the $\mathcal{C}$-Completion

For $\mathcal{A}$-dioids D, C, the centralizer of C in $D \otimes_{\mathcal{A}} C$ is

$$Z_C(D \otimes_{\mathcal{A}} C) := \{\varphi \in D \otimes_{\mathcal{A}} C \mid \varphi(1 \otimes c) = (1 \otimes c)\varphi \text{ for all } c \in C\}.$$

This is an $\mathcal{A}$-dioid, by properties of $\sum : \mathcal{A}(D \otimes_{\mathcal{A}} C) \to D \otimes_{\mathcal{A}} C$.

Lemma For $\mathcal{R}$-dioid K,

$$Z_{C_2'}(K \otimes_{\mathcal{R}} C_2') = \{[R] \mid R \in \mathcal{R}(K \times C_2'), R \subseteq K \times \{0, 1\}\}.$$

Theorem (Algebraic representation of $Q^{\mathcal{C}}_{\mathcal{R}}(\mathcal{R}M) = \mathcal{C}M$)

For each monoid M, $\mathcal{C}M \simeq Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2')$.

Proof: (using $M = X^*$)

1. For each $L \in \mathcal{C}M$, its elem.wise image has a least upper bound

$$\widehat{L} := \sum \{\{m\} \otimes 1 \mid m \in L\} \in Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2').$$

Prf.: For $L \in \mathcal{C}X^*$ there is $R \in \mathcal{R}(X^*[\Delta_n])$ with $L = h_{X^*}(R \cap D_X)$.

Code brackets p_i, q_i of Δ_n by the two pairs b, d and p, q of Δ_2 via

$$\overline{p_i} := bp^{i+1} \in P_2^* p, \qquad \overline{q_i} := q^{i+1} d \in qQ_2^*.$$

So in P_2': $\overline{p_i}\,\overline{q_j} = \delta_{i,j}$ and $b\overline{q_i} = 0 = \overline{p_i}d$, hence $b\,\overline{Q_n^*}\,\overline{P_n^*}\,d = \{0, 1\}$.

For $w \in X^*[\Delta_n]$:

$$w \in D_X \iff h_{\Delta_n^*}(w)/\rho_n = 1 \iff b\,\overline{h_{\Delta_n^*}(w)}\,d/\rho_2 = 1,$$
$$w \notin D_X \iff h_{\Delta_n^*}(w)/\rho_n \neq 1 \iff b\,\overline{h_{\Delta_n^*}(w)}\,d/\rho_2 = 0.$$

Let $h : X^*[\Delta_n] \to \mathcal{R}X^* \times \mathcal{R}\Delta_2^*/\rho_2$ be the homomorphism

$$w \mapsto (\{h_{X^*}(w)\}, \{\overline{h_{\Delta_n^*}(w)}\}/\rho_2)$$

Then $\mathcal{R}h$ maps $R \in \mathcal{R}(X^*[\Delta_n])$ to some $R' \in \mathcal{R}(\mathcal{R}X^* \times C_2')$, so

$$U := \{(\{1\}, \{b\}/\rho_2)\} \cdot R' \cdot \{(\{1\}, \{d\}/\rho_2)\} \in \mathcal{R}(\mathcal{R}X^* \times C_2'),$$

$$\begin{aligned} \text{and} \quad [U] &= \textstyle\sum\{\{h_{X^*}(w)\} \otimes \{b\,\overline{h_{\Delta_2^*}(w)}\,d\}/\rho_2 \mid w \in R\} \\ &= \textstyle\sum\{\{h_{X^*}(w)\} \otimes 1 \mid w \in R \cap D_X\} \quad (\{m\} \otimes 0 = 0) \\ &= \textstyle\sum\{\{m\} \otimes 1 \mid m \in L\} = \widehat{L}. \end{aligned}$$

Since $U \subseteq \mathcal{R}X^* \times \{0, 1\}$, by the Lemma, $[U] \in Z_{C_2'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_2')$.

Then show

2. (Algebraic CST) $\widehat{\cdot} : \mathcal{C}M \to Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2')$ is injective.

3. (Algebraic ReverseCST) The map $[R] \mapsto [R]^{\vee}$ given by

$$[R]^{\vee} := \bigcup \{A \mid (A,1) \in R\}, \quad \text{for } R \in \mathcal{R}(\mathcal{R}M \times C_2'),$$

is an injective map $\cdot^{\vee} : Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2') \to \mathcal{C}M$.

4. $\widehat{\cdot}$ and $\cdot^{\vee}$ are inverse to each other and homomorphisms.
5. $Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2')$ is a $\mathcal{C}$-dioid, $\widehat{\cdot}$ and $\cdot^{\vee}$ are $\mathcal{C}$-morphisms. $\square$

Each element of $\mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$ is the value of a regular expression $r \in \mathit{RegExp}(X \dot{\cup} \Delta_2)$ in the generators X and Δ_2 of $C_2' = \mathcal{R}\Delta_2^*/\rho_2$.

Theorem (Algebraic representation of $Q^{\mathcal{C}}_{\mathcal{R}}(K)$ for $K \in \mathbb{D}\mathcal{R}$)

For each *-continuous Kleene algebra K,

- $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ *is a $\mathcal{C}$-dioid, i.e. μ-continuous Chomsky algebra,*
- $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ *is the $\mathcal{C}$-completion $Q^{\mathcal{C}}_{\mathcal{R}}(K)$ of K.*

Corollary

For any $K \in \mathbb{D}\mathcal{R}$, there is a $\mathcal{C}$-morphism $\widehat{\cdot} : \mathcal{C}K \to Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$, $\widehat{L} := \sum\{m \otimes 1 \mid m \in L\}$, such that $\mathcal{C}K/_{\ker(\widehat{\cdot})} \simeq Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$.

The "categorical Chomsky-Schützenberger Theorem" is

$$Q^{\mathcal{C}}_{\mathcal{R}}(K) \lesssim Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'),$$

the analoge of the CST $\mathcal{C}X^* \subseteq \{h_{X^*}(R \cap D_X) \mid R \in \mathcal{R}(X^*[\Delta_2])\}$.

Application: Regular expressions for context-free languages

Let $L = \{a^n c b^n \mid n \in \mathbb{N}\} \in \mathcal{C}X^*$ and $\Delta_2 = \{\langle 0|, |0\rangle, \langle 1|, |1\rangle\}$.

For $w \in X^*$ and $t \in \Delta_2^*$ write wt for $\{w\} \otimes \{t\}/\rho_2 \in \mathcal{R}X^* \otimes_{\mathcal{R}} C_2'$.

$$
\begin{aligned}
\langle 0|(a\langle 1|)^* c(|1\rangle b)^*|0\rangle &= \sum_{n,m\in\mathbb{N}} \langle 0|(a\langle 1|)^n c(|1\rangle b)^m|0\rangle \quad (^*\text{-continuity}) \\
&= \sum_{n,m\in\mathbb{N}} a^n c b^m \underbrace{\langle 0|\langle 1|^n|1\rangle^m|0\rangle}_{\delta_{n,m}} \quad (x, t \text{ commute}) \\
&= \sum_{n\in\mathbb{N}} a^n c b^n \quad = \widehat{L} = [U] \quad \text{for}
\end{aligned}
$$

$$U = \{(\{a^n c b^m\}, \{\langle 0|\langle 1|^n|1\rangle^m|0\rangle\}/\rho_2) \mid n, m \in \mathbb{N}\} \in \mathcal{R}(\mathcal{R}X^* \times C_2').$$

Elements of $Z_{C_2'}(\mathcal{R}X^* \otimes_{\mathcal{R}} C_2')$ are those $\langle 0|r|0\rangle$ where r has $\langle 0|, |0\rangle$ only in codes $\overline{p_i} = \langle 0|\langle 1|^{i+1}$, $\overline{q_i} = |1\rangle^{i+1}|0\rangle$ of other brackets p_i, q_i.

CST-proof: $CF \ni G \mapsto r_G \in RegExp$ with $L(G) = h_{X^*}(L(r_G) \cap D_X)$

CF-grammar:

$$y \geq ayb + c$$

Least solution

$$L = \{a^n c b^n \mid n \in \mathbb{N}\} \in \mathcal{C}X^*$$

wrap rhs variables by brackets:

$$y \geq a\langle 1|y|1\rangle b + c$$

$$L_\Delta = \{(a\langle 1|)^n c(|1\rangle b)^n \mid n \in \mathbb{N}\}$$

add continuation variables y_F:

$$\begin{aligned} y &\geq a\langle 1|y|1\rangle b\, y_F + c\, y_F \\ y_F &\geq 1 \end{aligned}$$

$$L_\Delta \in \mathcal{C}(X^*[\Delta])$$

$$L = h_{X^*}(L_\Delta)$$

break into right-linear

initial- and follow-factors:

$$\begin{aligned} y &\geq a\langle 1|y + c\, y_F \\ y_F &\geq 1 + |1\rangle b\, y_F \end{aligned}$$

$$R = (a\langle 1|)^* c(|1\rangle b)^* \in \mathcal{R}(X^*[\Delta])$$

$$L_\Delta = R \cap D_X \qquad \square$$

We constructed a *-continuous Kleene algebra $K \supsetapprox \mathcal{C}X^*$ s.th.

$$L = \mu y(ayb + c)^{\mathcal{C}X^*} \simeq \langle 0|(a\langle 1|)^* c(|1\rangle b)^* |0\rangle^K.$$

Open Problems

1. Application in parser generation and compilation, e.g.
   - automata of $\langle 0|r_G|0\rangle$ for parsing with CFG G
   - μ-terms of depth n vs. $Dyck_n$, balanced brackets of depth n
2. Which $\mathbb{D}\mathcal{A}$ are closed under $n \times n$-matrix semiring formation ?
3. A similar algebra of "Turing-expressions" for $\mathcal{T}X^*$ in $\mathbb{D}\mathcal{T}$?
4. A context-sensitive cat. $\mathbb{D}\mathcal{S}$ with "non-erasing"-homorphisms?

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