Towards a Theory of Conversion Relations for Prefixed Units of Measure

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RAMiCS XX

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Agenda

1. Introduction
2. Unit Algebra
3. Conversion Relations
4. Conclusion

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Motivation: A Famous Disaster

Metric Math Mistake Muffed Mars Meteorology Mission

"Nov. 10, 1999: A disaster investigation board reports that NASA's Mars Climate Orbiter burned up in the Martian atmosphere because engineers failed to ***convert units*** *from English to metric. The peer review preliminary findings indicate that one team used English units (e.g. inches, feet and pounds) while the other used metric units for a key spacecraft operation."*

Cost: 328 M$

(Wired, 2010-11-10)

Bottom-Up Evaluation

- Solution is easy in theory:
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  - Current methodology does not specify units in code yet.
- Theoretical foundations are deficient.

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- There is no consensus on requirements:
  - Recent survey [MBBS20] identified 296 libraries and 95 tools (OSS only).
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  - Functionalities are not nearly pairwise equivalent.
  - Some are clearly unsound.
- Nearly all abstractions are **operational**:
  - Including ISO 80000
  - Prescriptive rules for notation, pronounciation, calculation
  - No method for objective justification
  - No distinction between
    - logical necessities,
    - contingent (historical) conventions,
    - outright idiosyncrasies

Our Contributions

- A novel **denotational** approach:
  - Compatible with, but orthogonal to [Ken96]
  - Algebraic–relational formal model of units of measure and their conversion
  - Semantics for *future* tools
  - Operational rules justified by deduction

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  - Algebraic–relational formal model of units of measure and their conversion
  - Semantics for *future* tools
  - Operational rules justified by deduction
- Epistemological cleanup:

Abstract	logical necessities
Parameterize	by contingent conventions
Rectify	outright idiosyncrasies

Theoretical Language

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- Not theory **of** R&A, but theory **by** R&A!

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4. The SI prefix families are geometric sequences (welcome *ronna-* and *quetta-* in 2022!), but cannot be written as powers of a generator.
5. Tools tend to specify conversion by way of one canonical unit per dimension. As a result, e.g., *newton-meter* $\leftrightsquigarrow$ *joule*, *gray* $\leftrightsquigarrow$ *sievert*, and even *revolutions-per-minute* $\leftrightsquigarrow$ *becquerel* end up convertible.

Committee Confusion

The 16th Conférence Générale des Poids et Mesures,

considering

- the effort made to introduce SI units into the field of ionizing radiations,
- the risk to human beings of an underestimated radiation dose, a risk that could result from a ***confusion*** between absorbed dose and dose equivalent,
- that the ***proliferation*** of special names represents a danger for the Système International d'Unités and must be avoided in every possible way, but that this rule can be broken when it is a matter of ***safeguarding*** human health,

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[CGPM16.5]

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The sievert is ***equal*** to the joule per kilogram.

[CGPM16.5]

Paper Confusion

To convert between measurements in different units of the same dimension, we must specify conversion factors between ***various*** units of that dimension. A natural place to keep this information is in the definition of a unit: each unit specifies how to convert measurements in that unit to measurements in ***any other*** defined unit (for the same dimension).

[All+04]

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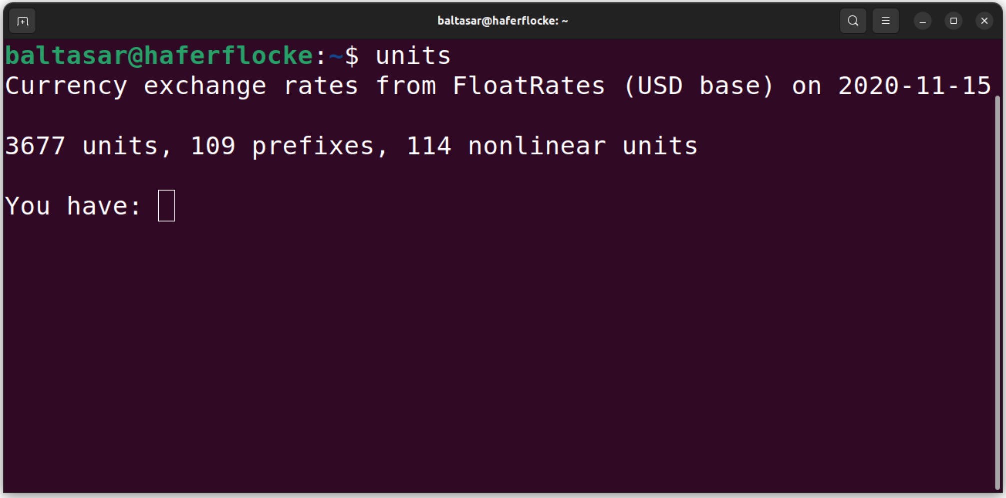
To convert between measurements in different units of the same dimension, we must specify conversion factors between ***various*** units of that dimension. A natural place to keep this information is in the definition of a unit: each unit specifies how to convert measurements in that unit to measurements in ***any other*** defined unit (for the same dimension).

Although the number of such conversion factors is quadratic in the number of units, it is not necessary to maintain so many factors explicitly: if we can convert between measurements in units A and B, and between measurements in units B and C, then we can convert between measurements in A and C via B. ***Thus***, it is sufficient to include in the definition of every unit a single conversion factor to a *primary unit* of that dimension, and convert between any two ***commensurable*** units via their common primary unit.

[All+04]

Tool Confusion



Tool Confusion

```
baltasar@haferflocke:~$ units
Currency exchange rates from FloatRates (USD base) on 2020-11-15

3677 units, 109 prefixes, 114 nonlinear units

You have: rpm
You want: becquerel
        * 0.10471976
        / 9.5492966
You have: 
```

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Semantic Sorts

Ratio e.g. $\frac{1000}{1}$, $\frac{5}{9}$, 568.26125, ...

Prefix e.g. k, M, G, ...

Unit e.g. m, N, Hz, ...

Dimension e.g. T, M, I, ...

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- Represented as finitely supported maps $X \to \mathbb{Z}$
  - Example: $\{a \mapsto 2, b \mapsto 1, c \mapsto -3\}_{/0}$
  - Group operation pointwise additive, ...
  - but commonly written as multiplicative, e.g. $a^2 \cdot b / c^3$
  - Danger of confusion: $1 = \varnothing_{/0} = a^0 b^0 c^0 \cdots$

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  - Monad multiplication $\lambda_X : \mathcal{U}\mathcal{A}^2(X) \to \mathcal{U}\mathcal{A}(X)$ — *flatten: e.g.* $\lambda\big((a^3)^2\big) = a^6$

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  - Pairing $\langle f_1, f_2 \rangle : G \to H_1 \times H_2$ from $f_i : G \to H_i$ — *e.g.* $\langle \mathrm{div}, \mathrm{mod} \rangle(7, 4) = (1, 3)$

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- Fixing G_1 yields a monad (monoid labeling)

$$\eta(x) = (1, x) \qquad \mu\big(a, (b, x)\big) = (ab, x)$$

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- $\mathcal{UA} \circ (G\times)$ is an accurate model of unit **syntax**.
  - Likely not a suitable monad – no group-friendly distributive law exists;
  - but maps naturally to the former by virtue of β.
  - Still useful: composite unit $\lfloor\ \rfloor = \delta\eta$

Big Picture

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P_b U_b D_b

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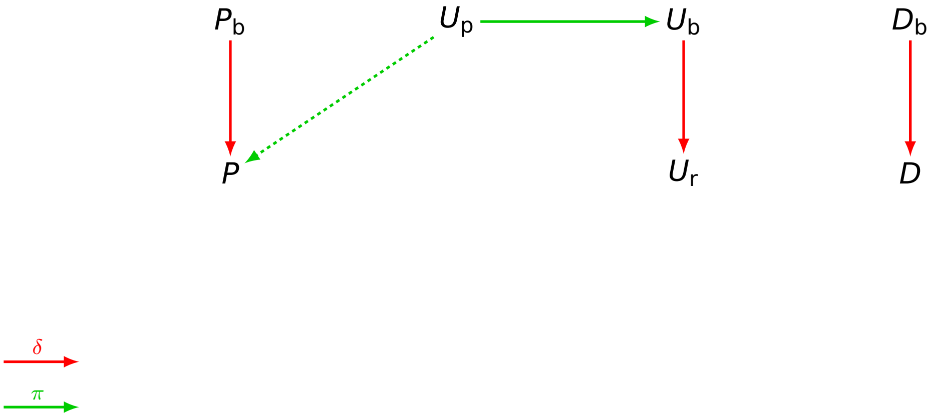

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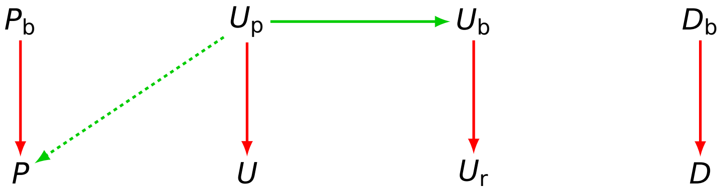

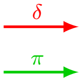

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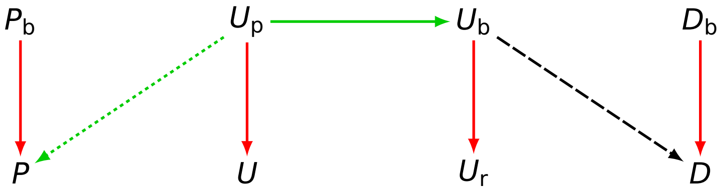

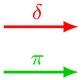

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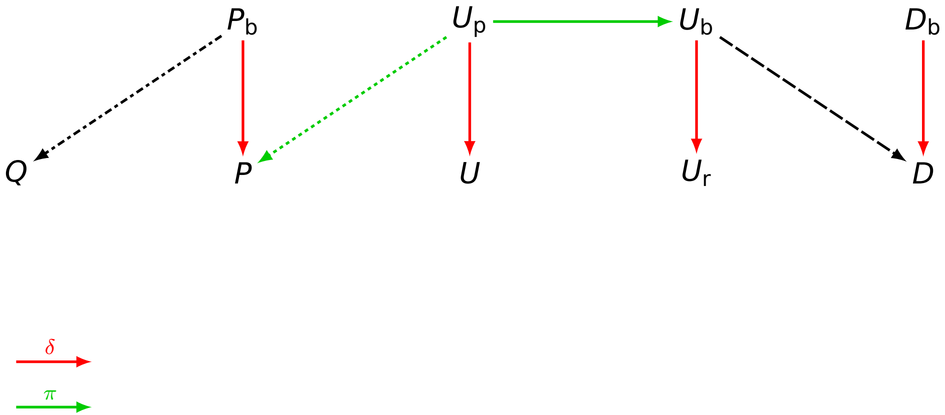

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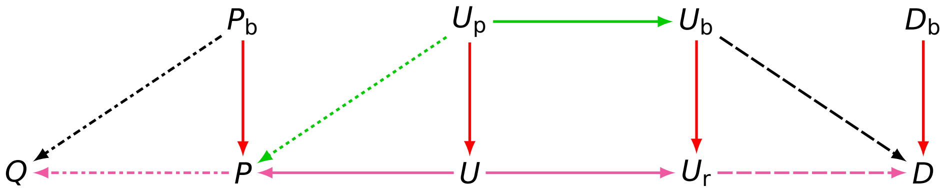

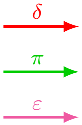

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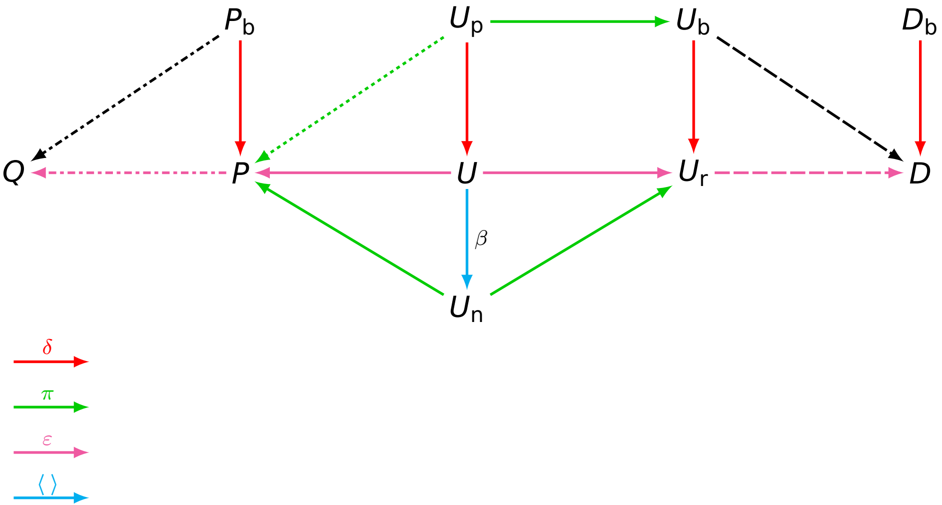

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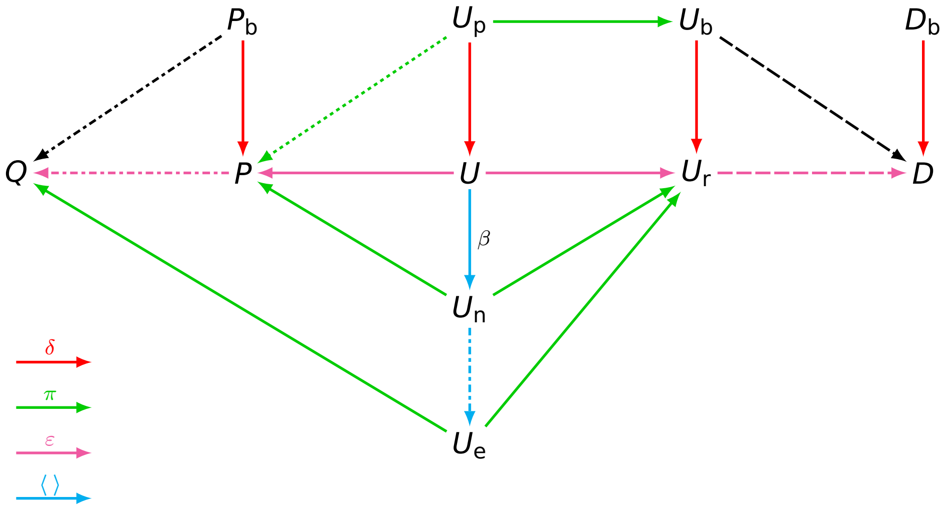

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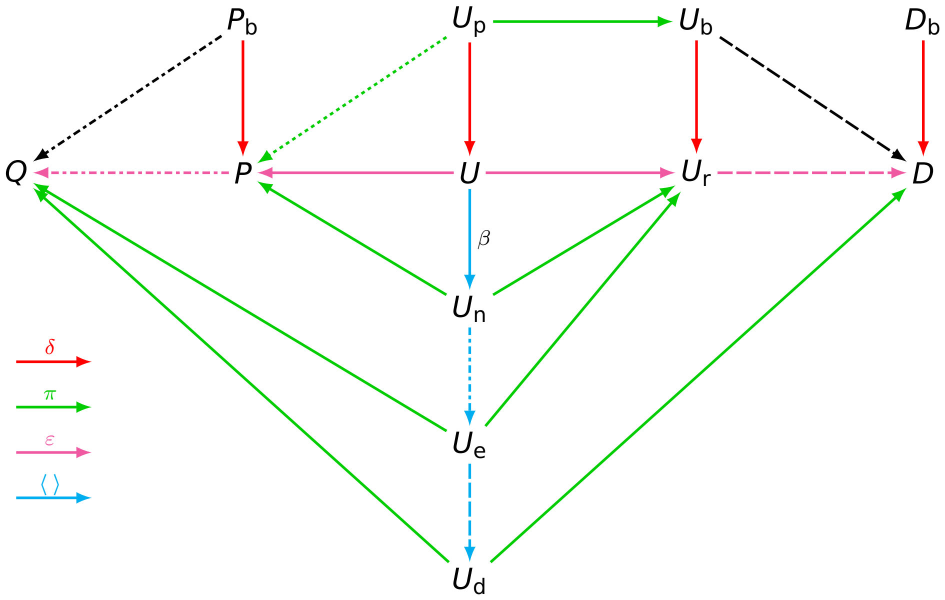

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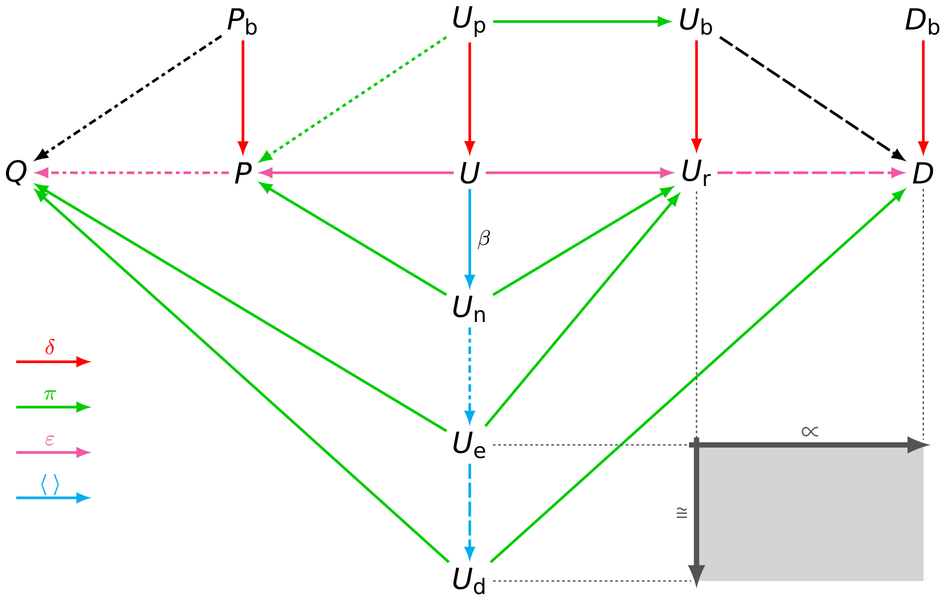

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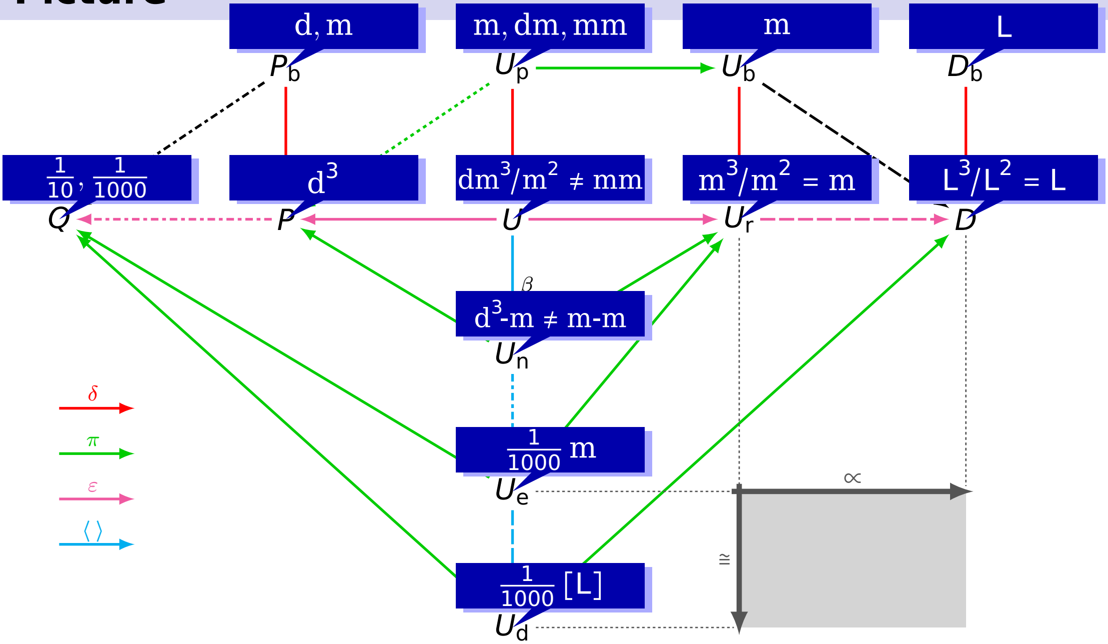
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$$C \subseteq U \times Q \times U$$

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2. FUNCTIONALITY $\quad (u, r, v), (u, r', v) \in C \implies r = r'$

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- Convertibility with factor 1 is called **coherence**:

$$u \propto_C v \iff u \xrightarrow{\exists r}_C v \qquad\qquad u \cong_C v \iff u \xrightarrow{1}_C v$$

Conversion Closure

- The **conversion closure** of a relation C of the above type is the smallest relation $C^* \supseteq C$ obeying three axioms:

3. MULTIPLICATION
$$u_1 \xrightarrow{r_1}_{C^*} v_1 \wedge u_2 \xrightarrow{r_2}_{C^*} v_2 \implies u_1 u_2 \xrightarrow{r_1 r_2}_{C^*} v_1 v_2$$

4. INVERSE
$$u \xrightarrow{r}_{C^*} v \implies u^{-1} \xrightarrow{r^{-1}}_{C^*} v^{-1}$$

5. DECOMPOSITION
$$u \xrightarrow{\mathrm{pval}(u)}_{C^*} \mathrm{strip}(u)$$

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4. Inverse
$$u \xrightarrow{r}_{C^*} v \implies u^{-1} \xrightarrow{r^{-1}}_{C^*} v^{-1}$$
5. Decomposition
$$u \xrightarrow{\mathrm{pval}(u)}_{C^*} \mathrm{strip}(u)$$

- The closure of a unit conversion is not necessarily a unit conversion:
  - Codimensionality is preserved, but contradictory factors $u \xrightarrow{r \neq r'}_{C^*} v$ can arise.
- Conversion closure lifts rewriting rules to compound units.

Special Cases

- A conversion is called **defining**, iff left-hand sides are basic and unique:

$$u \xrightarrow{r}_C v \quad \text{is of the form} \quad \lfloor u_0 \rfloor \xrightarrow{r}_C v$$

$$\lfloor u_0 \rfloor \xrightarrow{r}_C v \wedge \lfloor u_0 \rfloor \xrightarrow{r'}_C v' \implies v = v'$$

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  - Rewriting on U_e terminates after a bounded number of steps.
  - Definitions of SI units (e.g., ISO 80000-1) can be read as well-defining.

Conversion Hierarchy

- A conversion is called . . .

1. **consistent** iff its closure is again a conversion;
2. **closed** iff it is its own closure;
3. **finitely generated** iff it is the closure of a finite conversion;
4. **defined** iff it is the closure of a defining conversion;
5. **well-defined** iff it is the closure of a well-defining conversion;
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- Each property in the conversion hierarchy entails the preceding.

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- Every well-defining conversion is consistent, i.e., has a well-defined closure.
- For closed conversions, convertibility encodes the group word problem.
- For well-defined conversions, convertibility is computable efficiently by rewriting to a normal form.

Agenda

1 Introduction

2 Unit Algebra

3 Conversion Relations

4 **Conclusion**

Save the Mars Mission!

$$\lfloor \mathrm{N} \rfloor \xrightarrow{1} \delta(\mathrm{kg}) \lfloor \mathrm{m} \rfloor \lfloor \mathrm{s} \rfloor^{-2}$$

$$\lfloor \mathrm{lbf} \rfloor \xrightarrow{1} \lfloor \mathrm{lb} \rfloor g_{\mathrm{n}}$$

$$\lfloor \mathrm{lb} \rfloor \xrightarrow{a} \lfloor \mathrm{g} \rfloor$$

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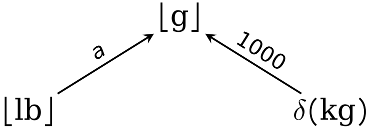

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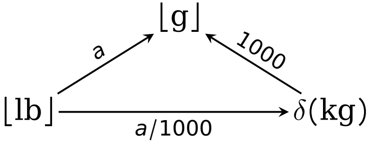

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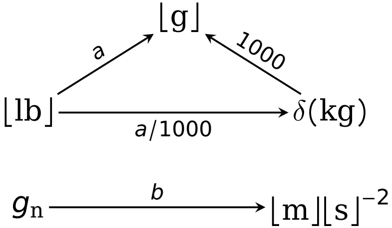

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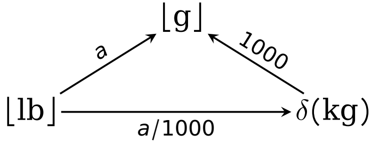

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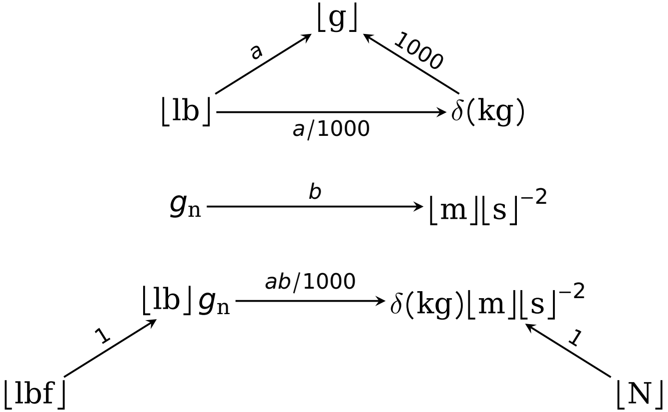

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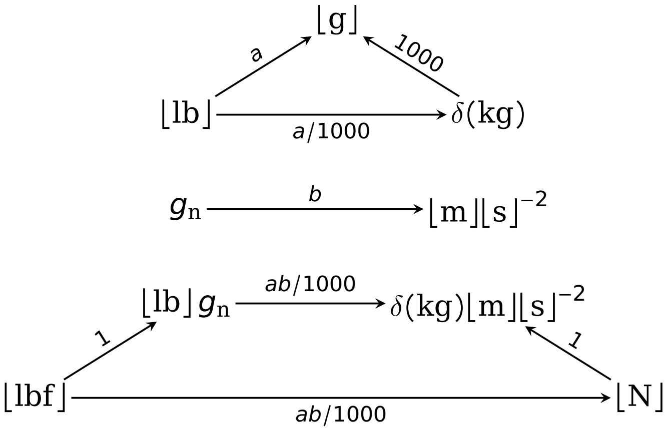

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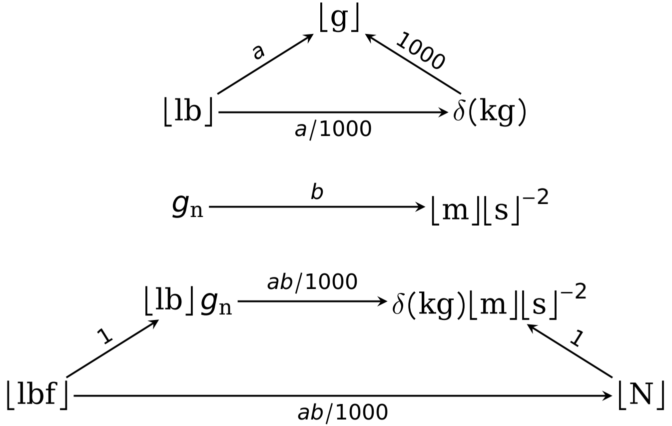

$$ab/1000 = 4.4482216152605$$

Example Symptoms, Revisited

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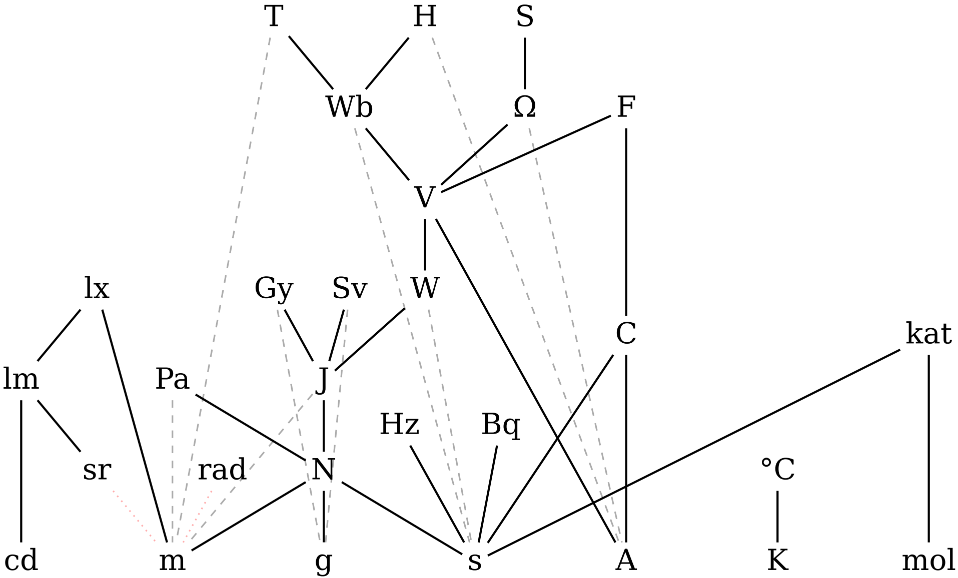
5. Tools tend to specify conversion by way of one canonical unit per dimension.
   - Conversion relations are (transitively) closed, yet allow multiple disconnected components per dimension.

Bonus Track: The 29 Named SI Units



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