Enumerating, Cataloguing and Classifying all Quantales on up to nine elements

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Agenda

- Definitions: Quantales
- Potential Applications of Quantales
- Enumerating the Quantales - Methodology
- Enumerating the Quantales - Results
- Discussion & Future Plans

- More ”Show & Tell” than ”Definition-Theorem-Proof”
- Very combinatorially flavored talk

Quantales

Definition: Quantale

A set X, and two binary operations $*, \vee : X \times X \to X$, such that

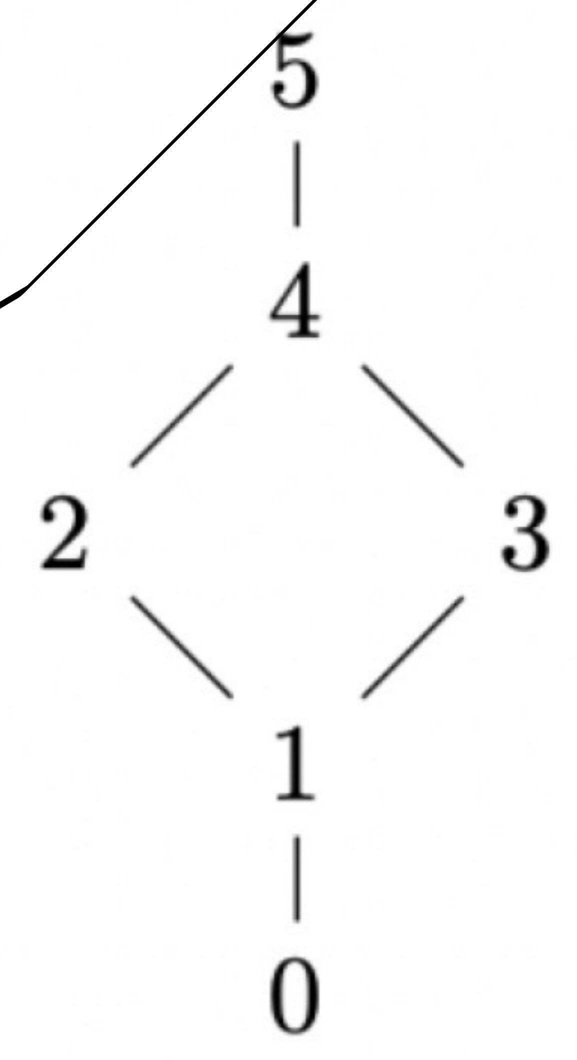
- $(X, *)$ is a semigroup
- $(X, \vee)$ is a join-lattice
- $*$ distributes over (arbitrary) $\vee$:

$$x * (y \vee z) = (x * y) \vee (x * z)$$

$$(x \vee y) * z = (x * z) \vee (y * z)$$

- (Finite case) $\bot = 0$ is a zero element under $*$

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	1	0	1	3
2	0	0	2	0	2	5
3	0	1	1	3	3	3
4	0	1	2	3	4	5
5	0	2	2	5	5	5



<u>Convention in this talk:</u>
Elements denoted 0,1,…,(N-1)
$\bot = 0,\ \top = (N-1)$, where N is the number of elements

Quantales

Since we study the finite case, we also have a uniquely determined infimum operator, $\wedge : X \times X \to X$.

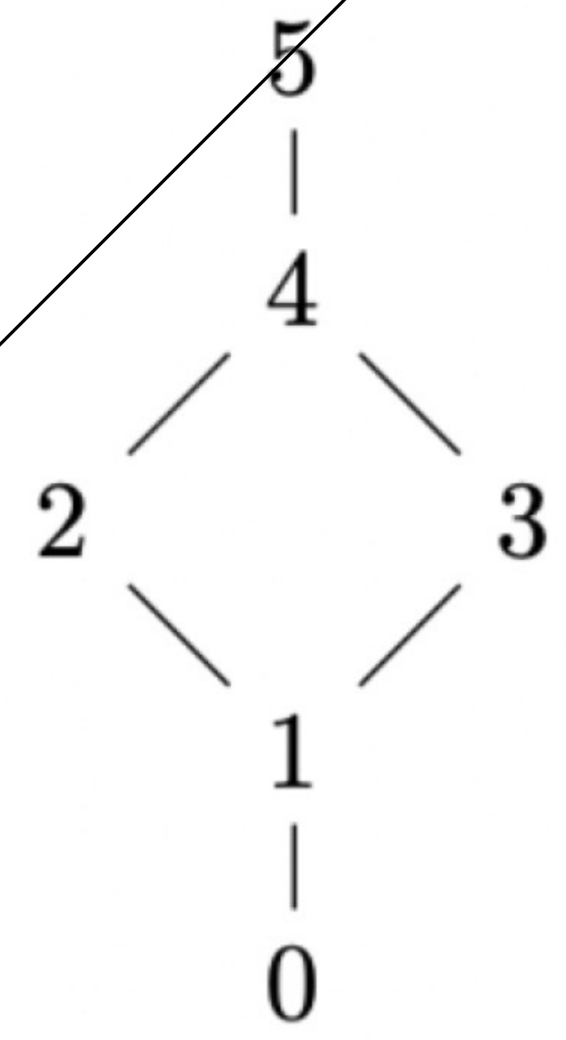
Additionally, every quantale has two uniquely determined adjoint operators, called *left* and *right implication / residuation*:

$\leftarrow, \to : X \to X$, s.t. $\forall x, y, z \in X$

- (Left) $x * y \leq z \Leftrightarrow x \leq y \leftarrow z$
- (Right) $x * y \leq z \Leftrightarrow y \leq x \to z$

Semantically, $z \leftarrow y$ is the maximal solution (interpreted in the lattice) of $x * y \leq z$ in x, and $x \to z$ is the maximal solution of $x * y \leq z$ in y.

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	1	0	1	3
2	0	0	2	0	2	5
3	0	1	1	3	3	3
4	0	1	2	3	4	5
5	0	2	2	5	5	5



←	0	1	2	3	4	5
0	5	5	5	5	5	5
1	2	4	5	4	5	5
2	0	3	5	3	5	5
3	2	2	2	4	4	5
4	0	1	2	3	4	5
5	0	0	0	3	3	5

→	0	1	2	3	4	5
0	5	5	5	5	5	5
1	3	4	4	5	5	5
2	3	3	4	3	4	5
3	0	2	2	5	5	5
4	0	1	2	3	4	5
5	0	0	2	0	2	5

Quantales

- If an element d satisfies

$$d \leftarrow (x \rightarrow d) = x = (d \leftarrow x) \rightarrow d$$

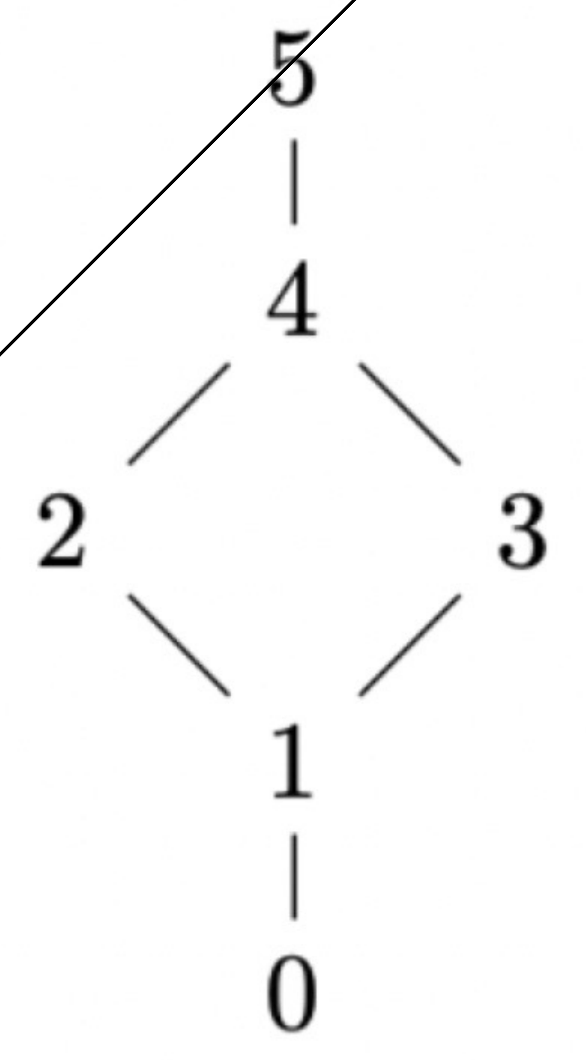
it is called *dualizing.*

- If an element d satisfies

$$x \rightarrow d = d \leftarrow x$$

it is called *cyclic.*

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	1	0	1	3
2	0	0	2	0	2	5
3	0	1	1	3	3	3
4	0	1	2	3	4	5
5	0	2	2	5	5	5



Properties
Unital
Balanced
Factor
Girard

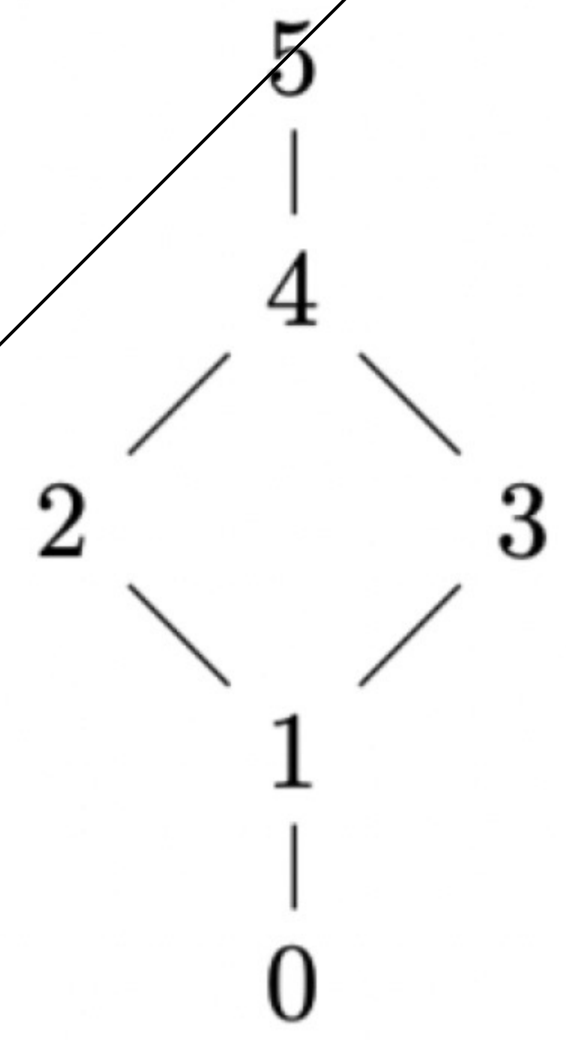
Special Elements
Cyclic elements: 1, 5
Dualizing elements: 1
Leftsided: 2
Rightsided: 3
Twosided: 0, 5

Quantales

Connection to Logic

- Elements →
  truth values of different degrees
- Semigroup operator →
  many-valued logical operator
- Residuations →
  many-valued logical implications
- Cyclic + dualizing elements →
  Linear Logic

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	1	0	1	3
2	0	0	2	0	2	5
3	0	1	1	3	3	3
4	0	1	2	3	4	5
5	0	2	2	5	5	5



Properties
Unital
Balanced
Factor
Girard

Special Elements
Cyclic elements: 1, 5
Dualizing elements: 1
Leftsided: 2
Rightsided: 3
Twosided: 0, 5

Quantales
Applications

- Models for many-valued logic, quantum logic, linear logic…

  Yetter, David N. *Quantales and (noncommutative) linear logic. The Journal of Symbolic Logic* 55.1 (1990): 41-64.

- Diagnosis systems

  P. Eklund, U. Höhle, J. Kortelainen, *Modules in health classifications*, 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Naples, 2017, pp. 1-6. DOI: 10.1109/FUZZ-IEEE.2017.8015430

- Circuit design

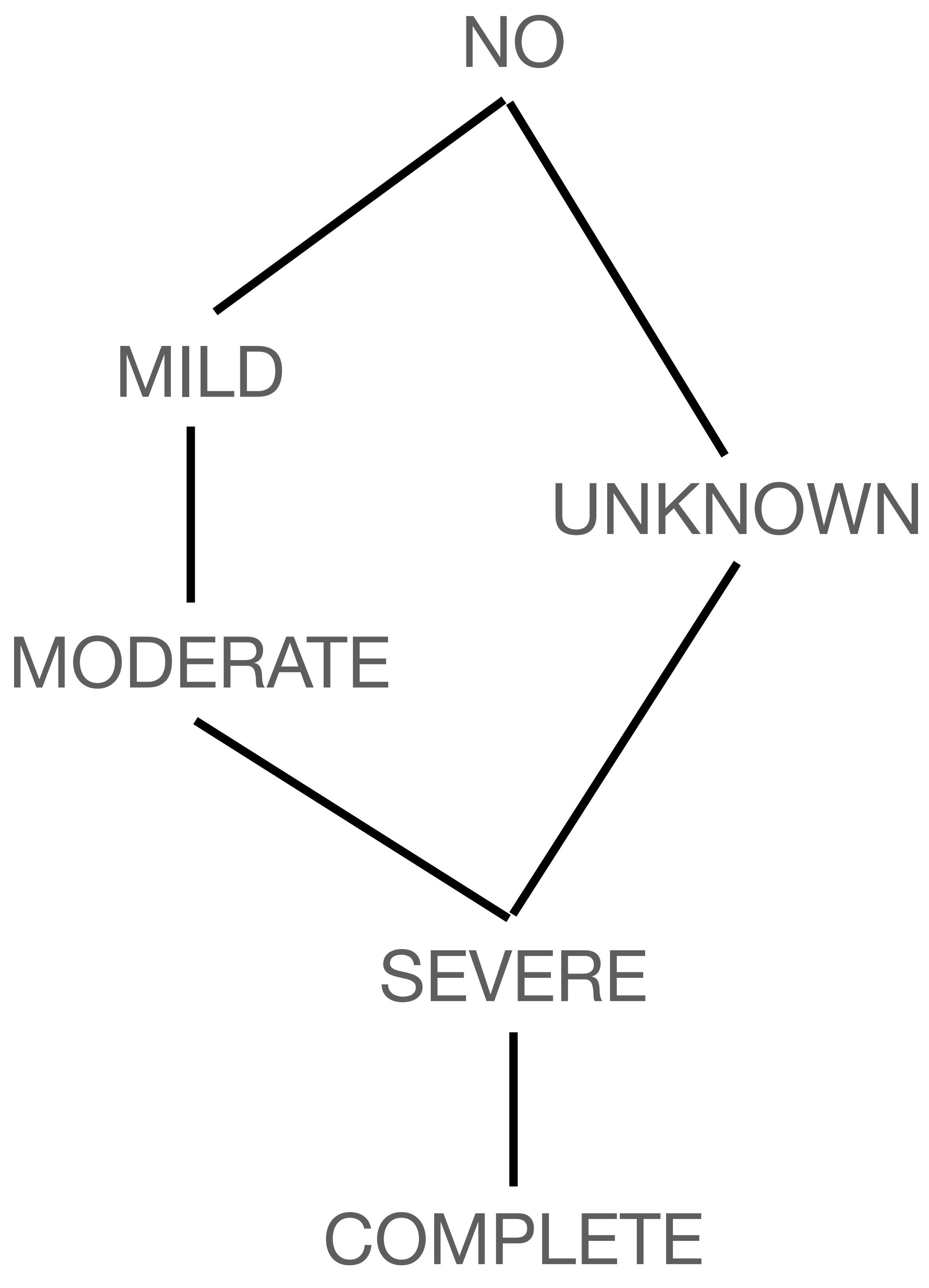
  P. Eklund, *Quantales in circuit design*, 2021 IEEE 51st International Symposium on Multiple-Valued Logic (ISMVL), 39–42.

- More to come?

Quantales

Applications - Diagnosis Systems

*	UNKNOWN	NO	MILD	MODERATE	SEVERE	COMPLETE
UNKNOWN	UNKNOWN	NO	MILD	MODERATE	SEVERE	COMPLETE
NO	NO	NO	MILD	MILD	SEVERE	COMPLETE
MILD	MILD	MILD	MILD	MILD	COMPLETE	COMPLETE
MODERATE	MODERATE	MILD	MILD	MILD	COMPLETE	COMPLETE
SEVERE	SEVERE	SEVERE	SEVERE	SEVERE	COMPLETE	COMPLETE
COMPLETE	COMPLETE	COMPLETE	COMPLETE	COMPLETE	COMPLETE	COMPLETE



Taken from Eklund, Patrik. "Information and Process in Health." Integrated Care and Fall Prevention in Active and Healthy Aging. IGI Global, 2021. 263-279.

Quantales

Why Enumerate Quantales?

- ”How many quantales are there?” - No OEIS entries before these efforts!
- Understanding the ”Design Space” of quantales
- ”Bottom-up” understanding of quantales
- Finding examples, patterns and numbers of potential interest
- Test bench for conjectures
- Sandbox for experimentation

OEIS Foundation Inc. (2023), The On-Line Encyclopedia of Integer Sequences, Published electronically at https://oeis.org

Quantales

Overview of Enumeration of Related Structures

- Semigroups: up to 9 elements (A0027851)
  Various authors, state-of-the-art A.Distler et al.

N	1	2	3	4	5	6	7	8	9
#	1	5	24	188	1 915	28 634	1 627 672	3 684 030 417	105 978 177 936 292

- Lattices: up to 20 elements (A006966)
  Various authors, see OEIS entry

N	1	2	3	4	5	6	7	8	9	10	…	20
#	1	1	1	2	5	15	53	222	1 078	5 994	…	23 003 059 864 006

- Residuated Lattices: up to 12 elements (No OEIS entry)
  Belohlavek and Vychodil

N	1	2	3	4	5	6	7	8	9	10	11	12
#	1	1	2	7	26	129	723	4 712	34 698	290 565	2 779 183	30 653 419

GLIOC - Logic is Lative

LINZ 2022
39th Linz Seminar on Fuzzy Set Theory
Many-Valued Logics: Theory and Applications
Bildungszentrum St. Magdalena, Linz, Austria
June 7-10, 2022

Quantales

A Brief History of the Enumeration of Finite Quantales

- ≤ 2018: 12 quantales on 3 elements: Brute Force
  (exercise 2.3.1 in *Semigroups in Complete Lattices*)
- 2019: Enumeration up to 6 elements: SAT solvers
  (A. Shamsgovara, P. Eklund, M. Winter, *A Catalogue of Finite Quantales*, GLIOC Notes, December 2019.)
- 2020: Enumeration up to 8 elements: Mace4
  (No separate publication)
- 2022: Enumeration up to 9 elements: Mace4
  (Arman Shamsgovara, *A catalogue of every quantale of order up to 9 (abstract)*, LINZ2022, 39th Linz Seminar on Fuzzy Set Theory, Linz, Austria.)

Finding all Finite Quantales

Method - SAT Solvers

- First attempt: SAT solvers
- Minisat, Lingeling, PLingeling, …
- Fix the number of elements, N
- Introduce variables $*_{xyz}$: "$x * y = z$", $\vee_{xyz}$: "$x \vee y = z$", $\qquad x, y, z = 1 \ldots N$
- Ground all quantifiers in the resulting atoms, e.g.

$$\forall x, y, z : x * \underbrace{(y \vee z)}_{a} = \underbrace{\underbrace{(x * y)}_{b} \vee \underbrace{(x * z)}_{c}}_{d} \quad \to \quad \bigwedge_{x,y,z,a,b,c,d} \left(\vee_{yza} \wedge *_{xyb} \wedge *_{xzc} \to (*_{xad} \leftrightarrow \vee_{bcd})\right)$$

Eén, Niklas, and Niklas Sörensson. "An extensible SAT-solver." International conference on theory and applications of satisfiability testing. Springer, Berlin, Heidelberg, 2003.

Biere, Armin. "Lingeling, Plingeling and Treengeling entering the SAT competition 2013." *Proceedings of SAT competition* 2013 (2013): 1.

Finding all Finite Quantales

Method - SAT Solvers

- Convert resulting formulas to CNF $\rightarrow$ solve using SAT solver of your choice
- Decode the satisfying assignment $\rightarrow$ obtain tables for $*$ and $\vee$
- To find more, add a clause that ”blocks” the found assignment and repeat until UNSAT
- To avoid isomorphic duplicates, add permutations of the blocking clause too
- **Problem:**
  - Input file grows quickly as blocking clauses are added
  - Have to run the solver once for each quantale

Finding all Finite Quantales

Method - Example SAT input file

```
p cnf 192 287478
116 0
80 0
65 0
65 0
129 0
…
-192 -128 -128 -192 128 0
-192 -128 -128 -128 192 0
-192 -128 -128 -192 128 0
-192 -128 -128 -128 192 0
```

Finding all Finite Quantales

Method - Mace4

- Second attempt: Mace4
- ”Models And CounterExamples”
- Easy syntax for expressing algebraic axioms
- Can find all models in one run
- Comes with companion program to get rid of isomorphic duplicates (isofilter)
- In our experience, a faster and easier alternative to our SAT solver approach

```
assign(domain_size, 6).
assign(max_models, -1).
formulas(theory).

(x v y) v z = x v (y v z).
x v y = y v x.
x v x = x.

(x*y)*z = x*(y*z).
x*(y v z) = (x*y) v (x*z).
(x v y)*z = (x*z) v (y*z).

x v 0 = x.
x v 5 = 5.
0*x = 0.
x*0 = 0.

end_of_list.
```


W.McCune,”Prover9andMace4”,http://www.cs.unm.edu/mccune/Prover9,2005-2010.

Finding all Finite Quantales

Method - Mace4

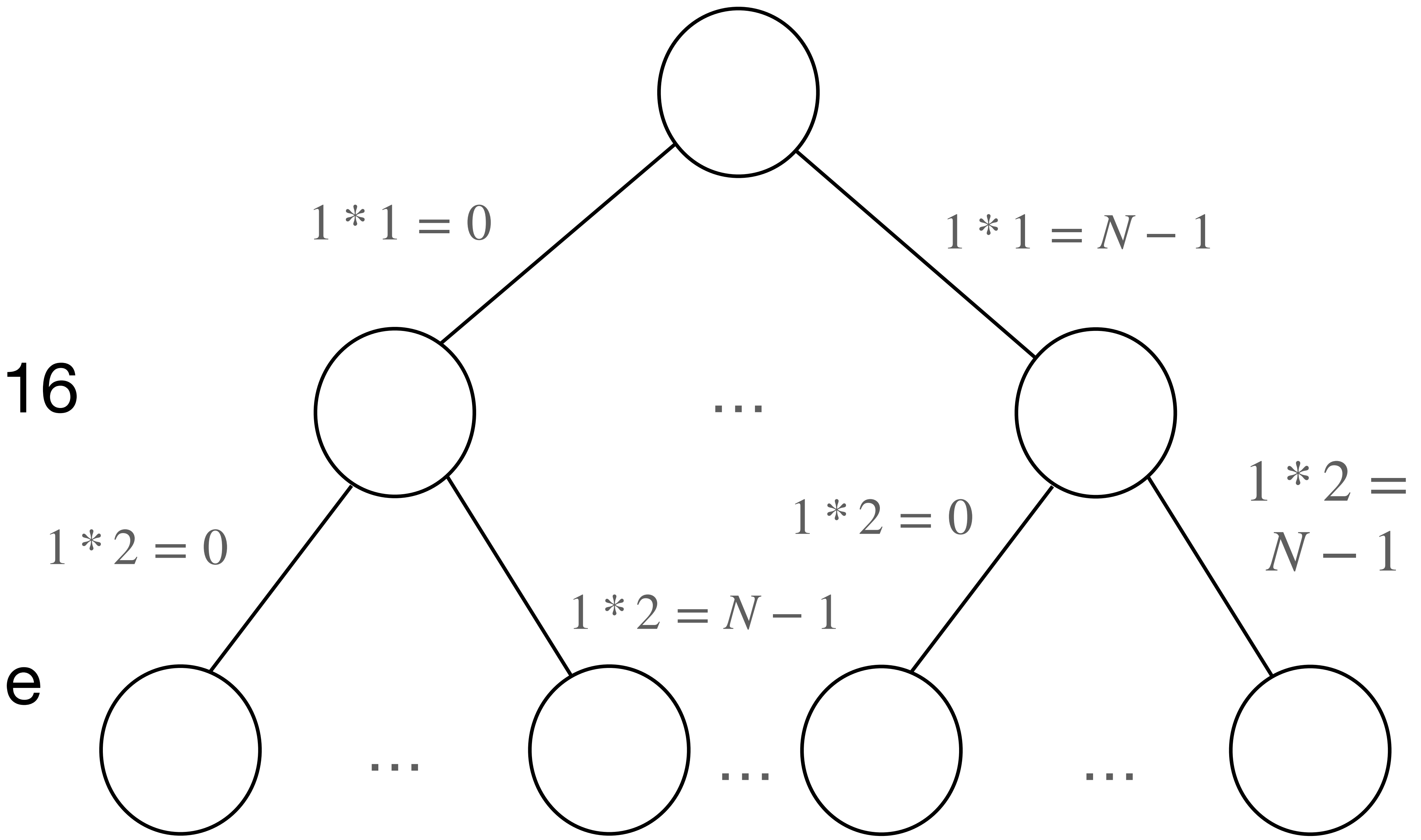
Problem:
Memory-related crashes if too many structures found

Band-aid:
Run on computer with a lot of RAM (≥32 GB works fine, ≤16 less so)

Better solution:
Branch searches, solve branches separately, then combine

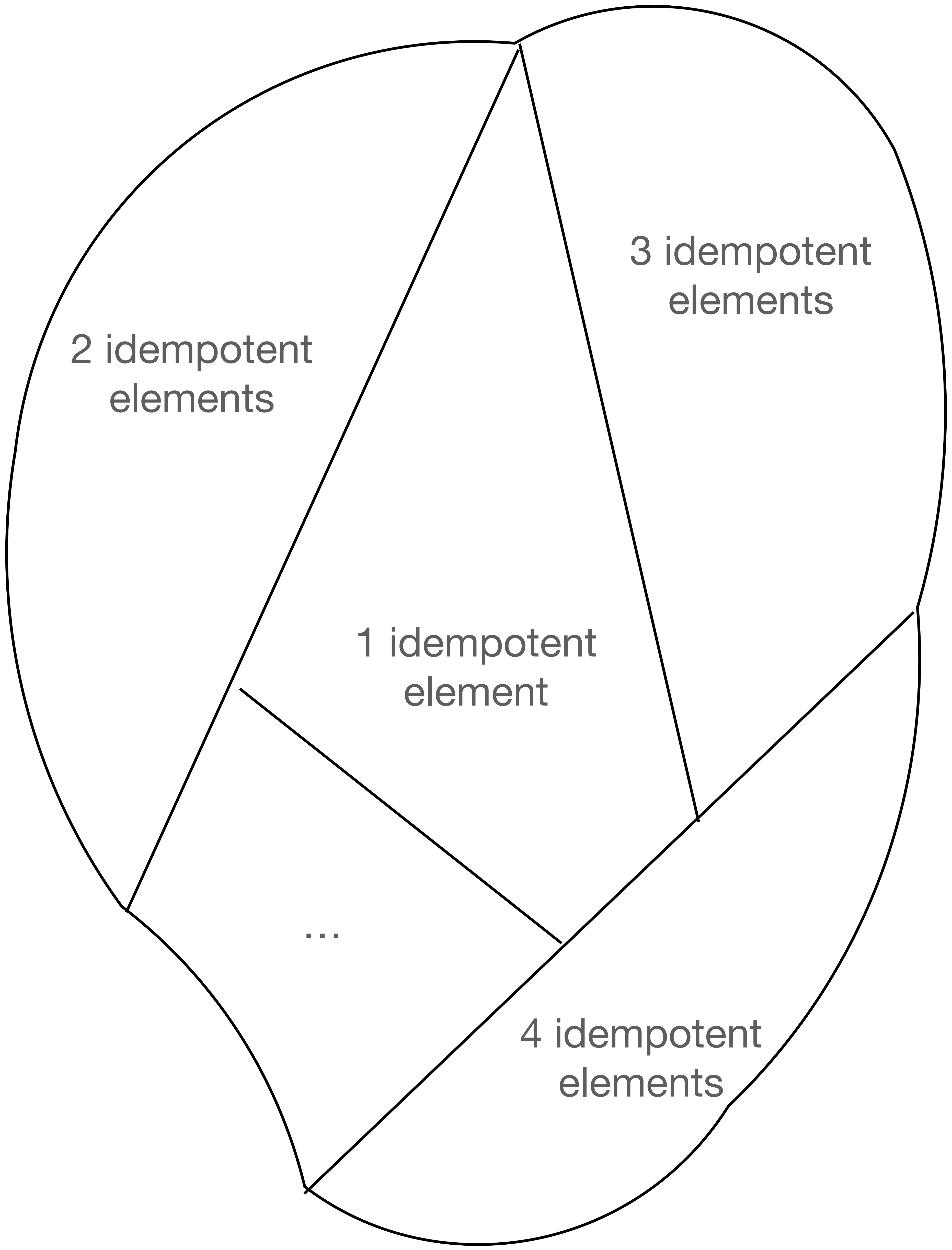
Problem:
Software guarantees that quantales found within a branch are non-isomorphic, but no a priori guarantees *between* branches

This means we can't e.g. pick a variable to branch on.



Finding all Finite Quantales
Method

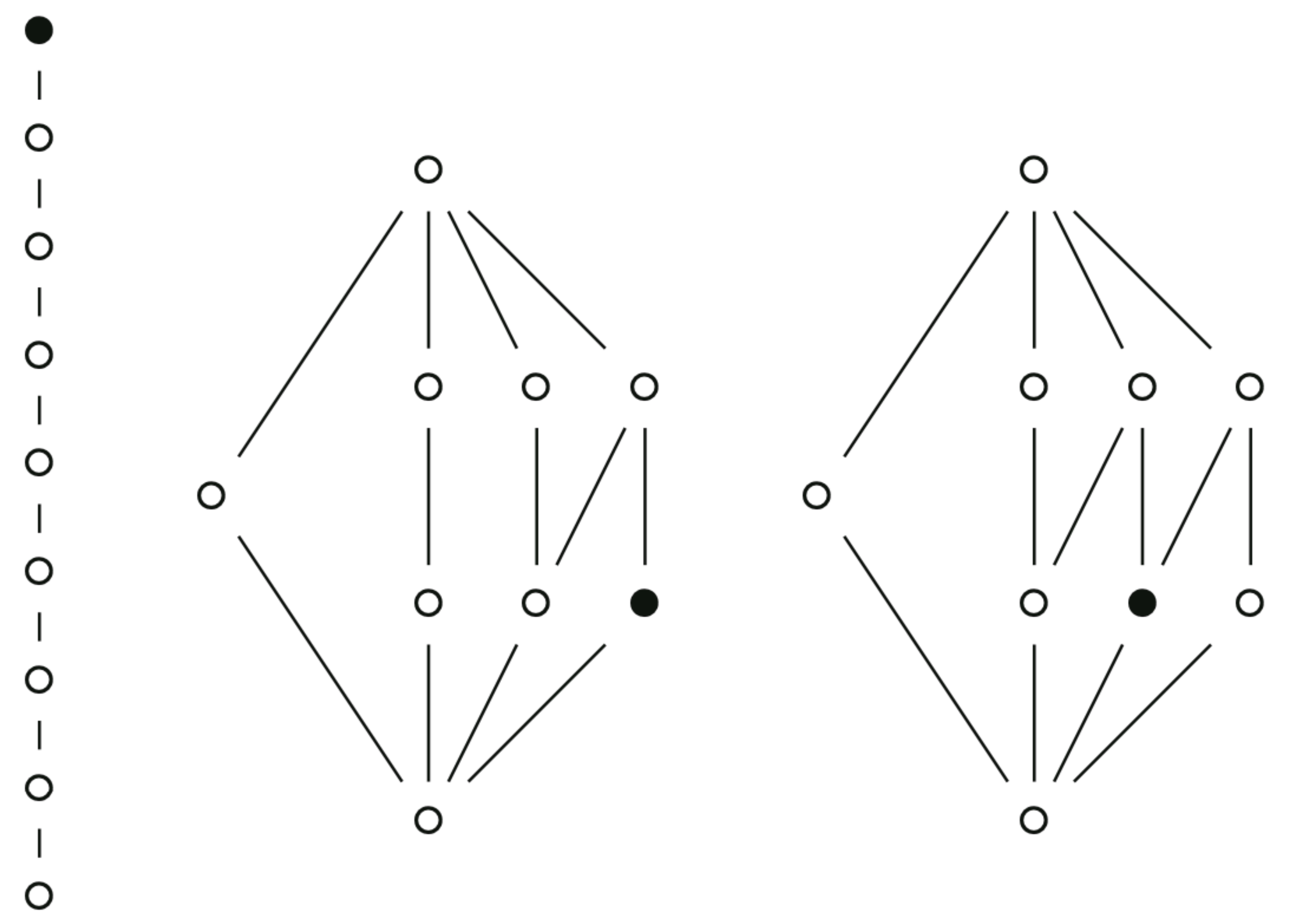
- Idea: Branch on properties that are invariant under isomorphism!
- Branching scheme, in order of use:
   1. Lattice table
   2. # Idempotent elements
   3. # Leftsided and # Rightsided elements
   4. # Elements x s.t. $x^2 = \top$
   5. Tailor-made branchings based on lattice
- Our experience: Avoid branchings with more than 1 quantified parameter (e.g. # commutative pairs)



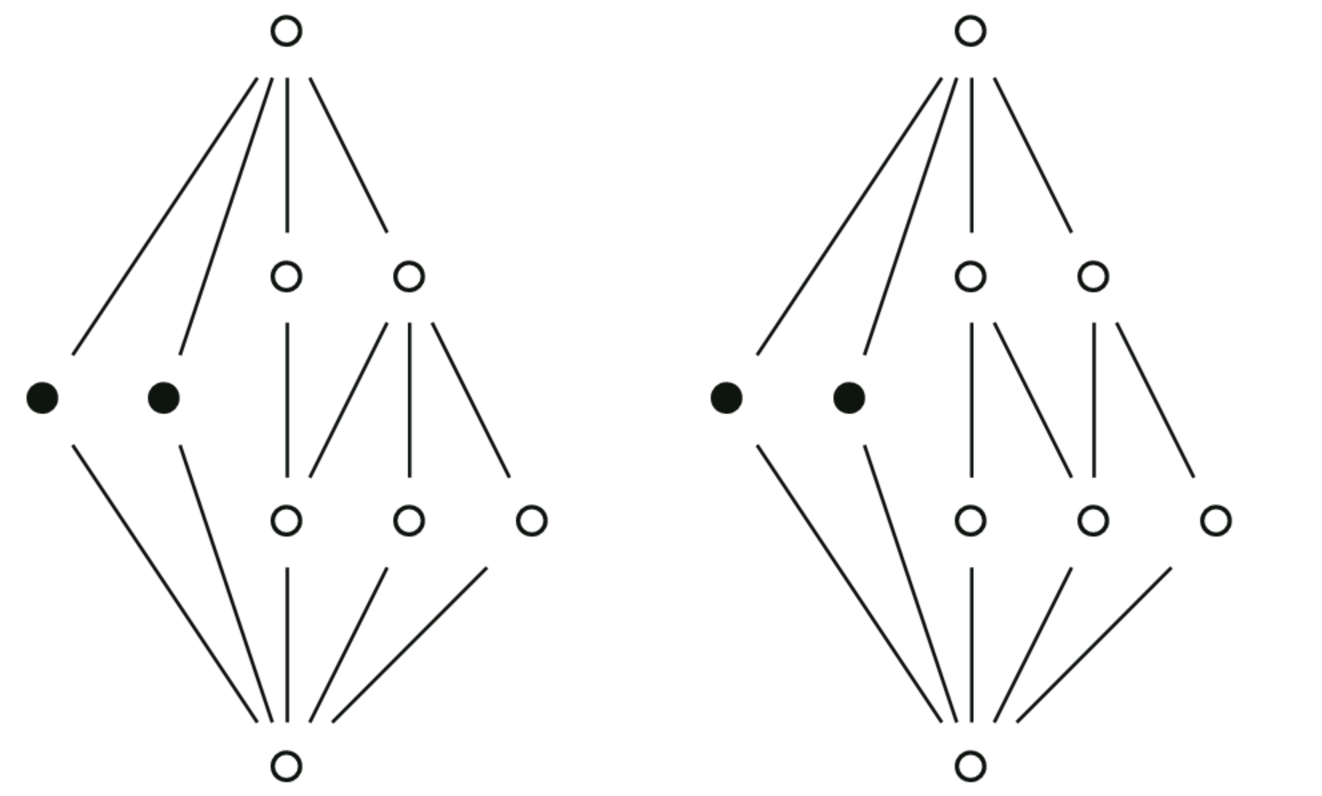
Finding all Finite Quantales

Method - Tailor-made branchings

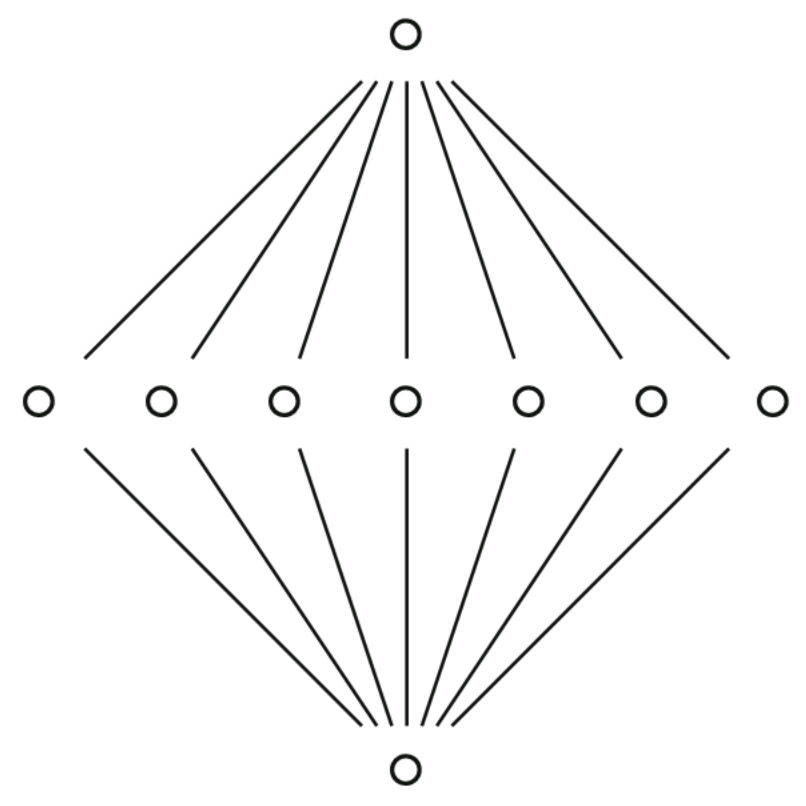
Branch on the value of x^2, x being the filled circle



Branch on whether $x^2 = \top$ holds for 0, 1 or 2 of the filled circles



Branch on the # middle elements x s.t. $x * y = \top$ for all y



Results

Fun Facts

- Can enumerate quantales on (very anecdotal measurements, assuming access to cpu:s with high core counts and ca. 50 parallel mace4 searches at all times):

$\leq$ 6 elements ”instantly”	(ca. 11 MB storage)
$\leq$ 7 elements within a minute	(ca. 286 MB storage)
$\leq$ 8 elements within a day	(ca. 9 GB storage)
$\leq$ 9 elements within a couple of months	(ca. 388 GB storage)

- Storing each quantale as two tables $*$, $\vee$, the full data set takes $\approx$ 400 GB storage space!
- The quantales on the 9-chain alone take up 17GB!
- Over 100 000 Mace4 input files in total

Results

A354493 Number of **quantales** on n elements, up to isomorphism. +20 7

1, 2, 12, 129, 1852, 33391, 729629, 19174600, 658343783 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS A **quantale** is an algebraic structure (X,*,v) composed of a set X of elements, a semigroup operator "*" and a supremum operator "v" (in the sense of lattices) such that * distributes over v: x * (y v z) = (x * y) v (x * z) and (x v y) * z = (x * z) v (y * z) for all elements x,y,z in X. In addition the bottom element corresponding to v, denoted 0, must satisfy x * 0 = 0 * x = 0.

REFERENCES

P. Eklund, J. G. García, U. Höhle, and J. Kortelainen, (2018). Semigroups in complete lattices. In Developments in Mathematics (Vol. 54). Springer Cham.

K. I. Rosenthal, **Quantales** and their applications. Longman Scientific and Technical, 1990.

Arman Shamsgovara, A catalogue of every **quantale** of order up to 9 (abstract, to appear), LINZ2022, 39th Linz Seminar on Fuzzy Set Theory, Linz, Austria.

Arman Shamsgovara and P. Eklund, A Catalogue of Finite **Quantales**, GLIOC Notes, December 2019.

Results

Number of Quantales on N elements

N	1	2	3	4	5	6	7	8	9
Quantales	1	2	12	129	1 852	33 391	729 629	19 174 600	658 343 783
Semi-unital	1	1	6	64	939	17 578	403 060	11 327 795	440 735 463
Unital	1	1	3	20	149	1 488	18 554	295 292	6 105 814
Left/Right-sided	1	2	9	60	497	4 968	58 507	807 338	13 341 730
Str Left/Right-sided	1	1	4	23	164	1 482	15 838	197 262	2 830 649
Two-sided	1	2	8	47	354	3 277	36 506	490 983	8 301 353
Integral	1	1	2	9	49	364	3 335	37 026	496 241
Balanced	1	1	9	106	1 597	29 720	663 897	17 747 907	620 659 554
Spatial	1	2	10	71	570	5 147	51 248	557 143	6 557 759
Str. Spatial	1	1	4	21	121	818	6 236	52 919	498 046
Idempotent	1	1	4	24	169	1 404	13 104	134 464	1 492 598

Results

Number of Quantales on N elements

N	1	2	3	4	5	6	7	8	9
Quantales	1	2	12	129	1 852	33 391	729 629	19 174 600	658 343 783
Semi-integral	1	2	11	96	1 041	13 669	211 561	3 780 964	77 057 208
Commutative	1	2	8	57	550	6 639	96 264	1 639 905	32 781 241
Bisymmetric	1	2	12	125	1 691	28 249	565 046	13 553 879	448 314 086
Factor	1	2	4	38	519	9 442	219 222	6 538 004	296 594 240
Prime	1	2	10	70	559	4 989	49 154	514 146	6 181 882
Str. Prime	1	1	4	20	115	764	5 749	48 255	450 342
Inf-distributuve	1	2	12	108	1 124	13 256	172 535	2 452 680	38 098 425
Comp. distr.	1	2	12	129	1 437	19 047	269 739	4 207 822	132 177 828
Frobenius	1	1	2	8	19	91	267	1 347	4 881
Girard	1	1	2	8	19	91	262	1 318	4 710

Results

Number of Quantales on the N element chain

N	1	2	3	4	5	6	7	8	9
Quantales	1	2	12	101	1 003	11 329	142 094	1 957 183	29 634 185
Semi-unital	1	1	6	45	414	4 324	49 997	631 949	8 681 521
Unital	1	1	3	15	84	575	4 687	45 223	516 882
Left/Right-sided	1	2	9	55	413	3 728	39 627	492 535	7 308 241
Str Left/Right-sided	1	1	4	20	133	1 087	10 512	118 112	1 527 872
Two-sided	1	2	8	44	308	2 641	27 120	332 507	5 035 455
Integral	1	1	2	8	44	308	2 641	27 120	332 507
Balanced	1	1	9	82	846	9 774	124 258	1 720 426	25 819 824
Spatial	1	2	10	55	293	1 536	8 007	41 663	216 626
Str. Spatial	1	1	4	14	48	164	560	1 912	6 528
Idempotent	1	1	4	17	82	422	2 274	12 665	72 326

Results

Number of Quantales on the N element chain

N	1	2	3	4	5	6	7	8	9
Quantales	1	2	12	101	1 003	11 329	142 094	1 957 183	29 634 185
Semi-integral	1	2	11	79	661	6 487	73 605	954 581	14 220 741
Commutative	1	2	8	41	241	1 553	10 704	77 811	591 441
Bisymmetric	1	2	12	97	877	8 677	92 268	1 047 921	12 933 247
Factor	1	2	4	24	187	1 737	18 423	218 026	2 846 283
Prime	1	2	10	55	293	1 536	8 007	41 663	216 626
Str. Prime	1	1	4	14	48	164	560	1 912	6 528
Inf-distributuve	1	2	12	101	1 003	11 329	142 094	1 957 183	29 634 185
Comp. distr.	1	2	12	101	1 003	11 329	142 094	1 957 183	29 634 185
Frobenius	1	1	2	4	8	17	38	91	222
Girard	1	1	2	4	8	17	38	91	222

Results

Number of Quantales on the diamond with N middle elements

N	3	4	5	6	7	8	9
Quantales	12	28	78	262	1 036	5 129	48 299
Semi-unital	6	19	67	249	1 021	5 112	48 280
Unital	3	5	8	17	42	176	1 421
Left/Right-sided	9	5	3	3	3	3	3
Str Left/Right-sided	4	3	2	2	2	2	2
Two-sided	8	3	1	1	1	1	1
Integral	2	1	0	0	0	0	0
Balanced	9	24	74	258	1 032	5 125	48 295
Spatial	10	16	19	23	27	31	35
Str. Spatial	4	7	8	10	12	14	16
Idempotent	4	7	9	11	13	15	17

Results

Number of Quantales on the diamond with N middle elements

N	3	4	5	6	7	8	9
Quantales	12	28	78	262	1 036	5 129	48 299
Semi-integral	11	17	21	25	29	33	37
Commutative	8	16	35	89	240	696	2 244
Bisymmetric	12	28	77	243	869	3 966	40 351
Factor	4	14	49	187	772	4 053	42 192
Prime	10	15	19	23	27	31	35
Str. Prime	4	6	8	10	12	14	16
Inf-distributuve	12	7	1	1	1	1	1
Comp. distr.	12	28	0	0	0	0	0
Frobenius	2	4	5	10	14	30	45
Girard	2	4	5	10	13	25	34

Results

Miscellaneous insights from the data

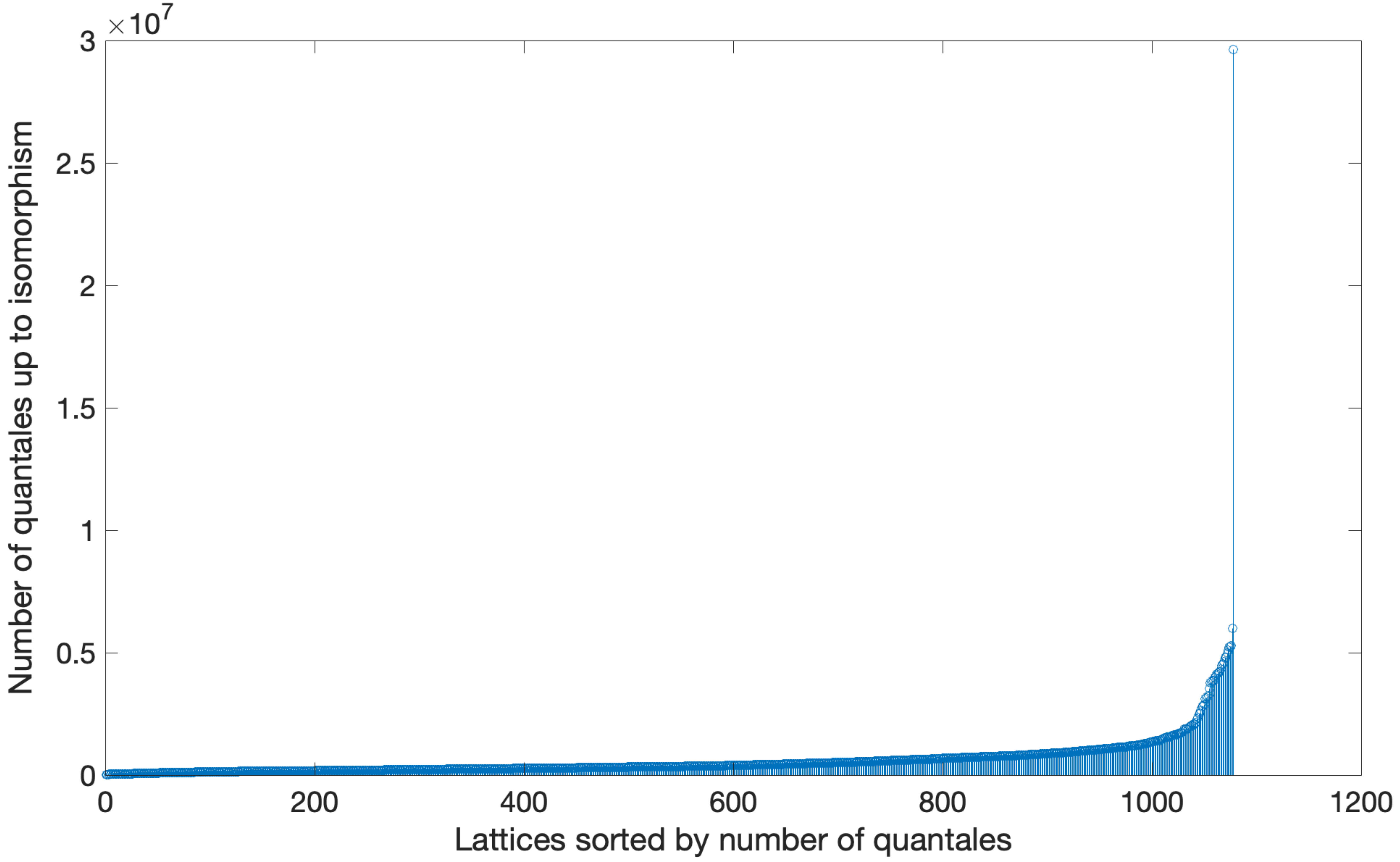
- Some pairs of semigroups and lattices can form multiple non-isomorphic quantales by ”permuting” one of them while keeping the other fixed.

- This intuitively explains why there are sometimes *more* quantales than semigroups.

N	1	2	3	4	5	6	7	8	9
Semigroups	1	5	24	188	1 915	28 634	1 627 672	3 684 030 417	105 978 177 936 292
Quantales	1	2	12	129	1 852	33 391	729 629	19 174 600	658 343 783
Lattices	1	1	1	2	5	15	53	222	1 078

Results

How do the quantales distribute over lattices?

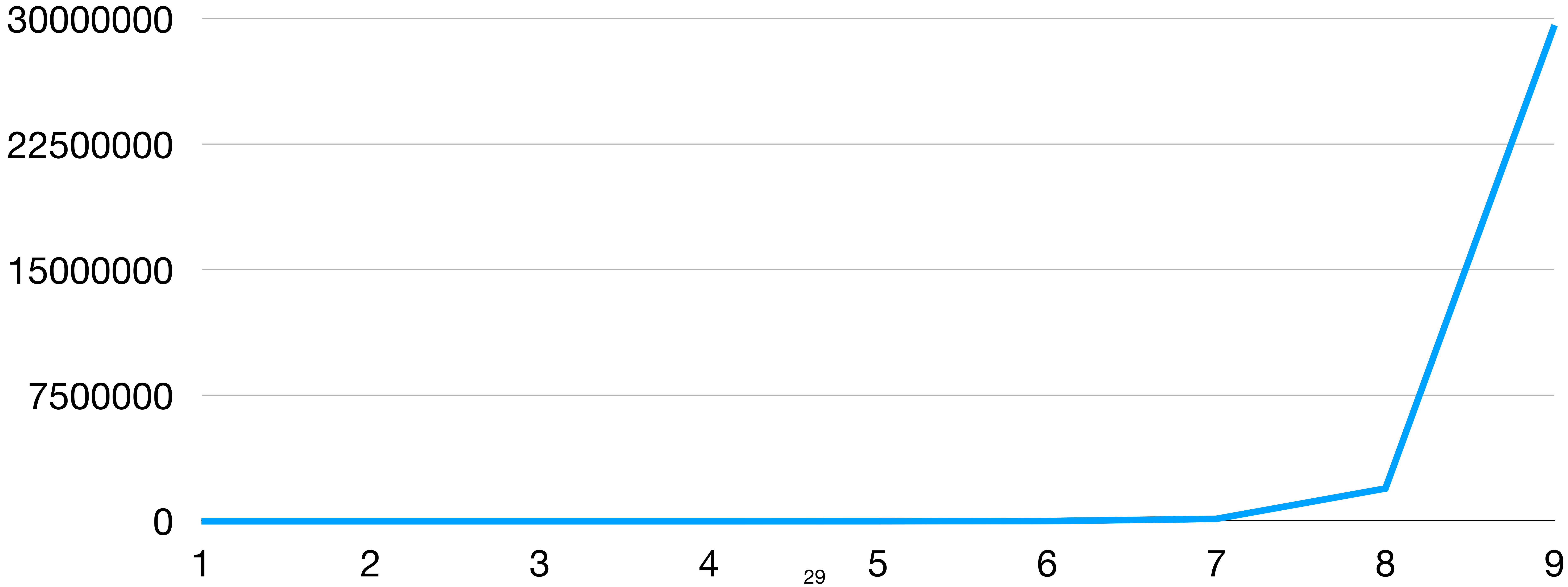


Results

Which lattice has the most quantales?

(Answer: the chain lattice)

N	1	2	3	4	5	6	7	8	9
Quantales on N-chain	1	2	12	101	1 003	11 329	142 094	1 957 183	29 634 185

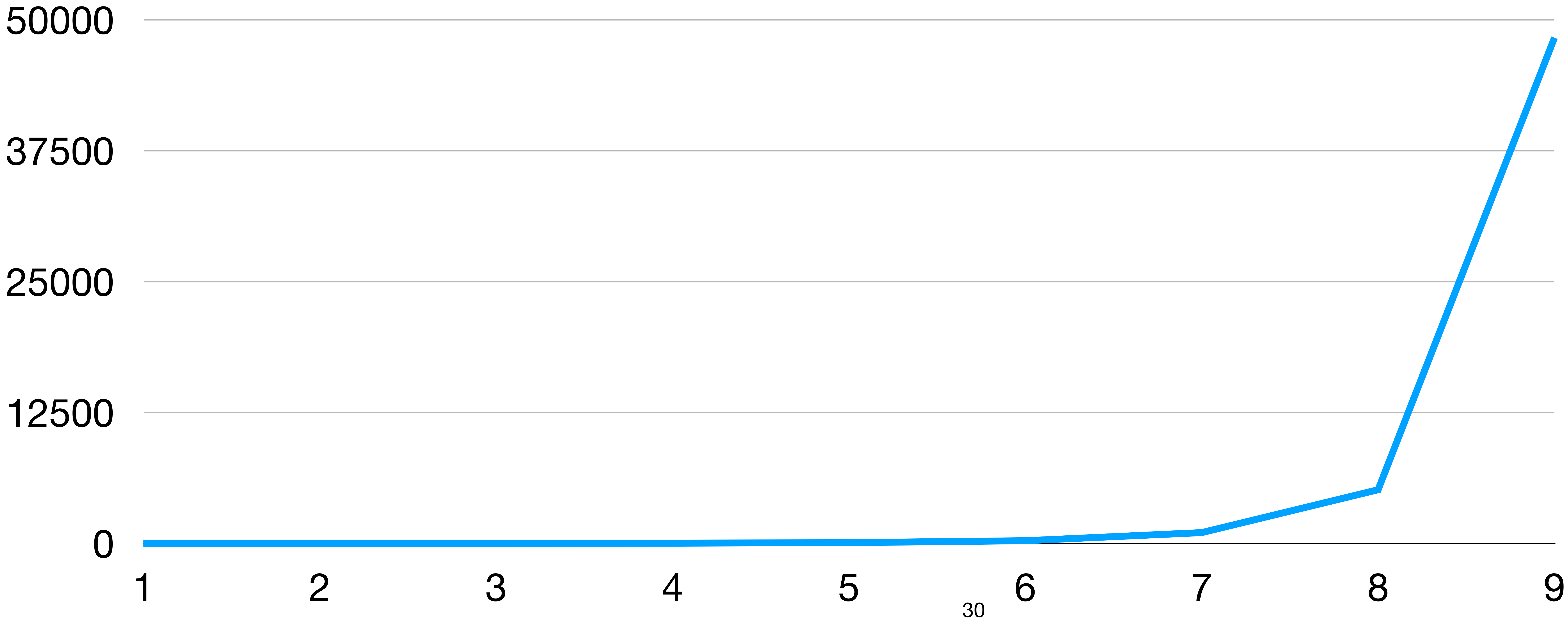


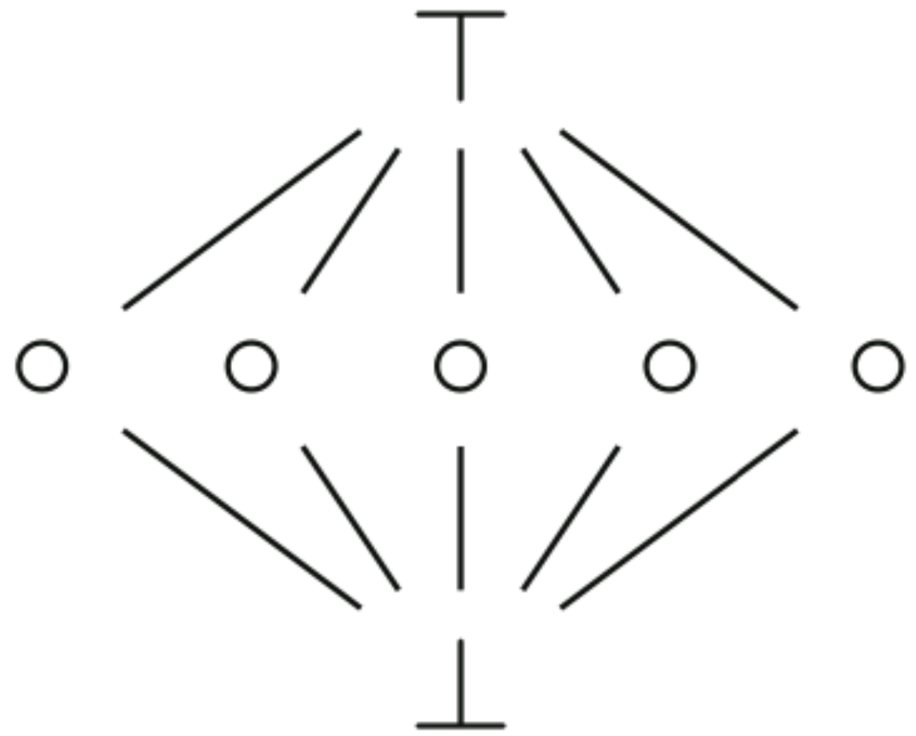
Results

Which lattice has the least quantales?

(Answer: the ”diamond” lattice…?)

N	1	2	3	4	5	6	7	8	9
Quantales on diamond	1	2	12	28	78	262	1 036	5 129	48 299

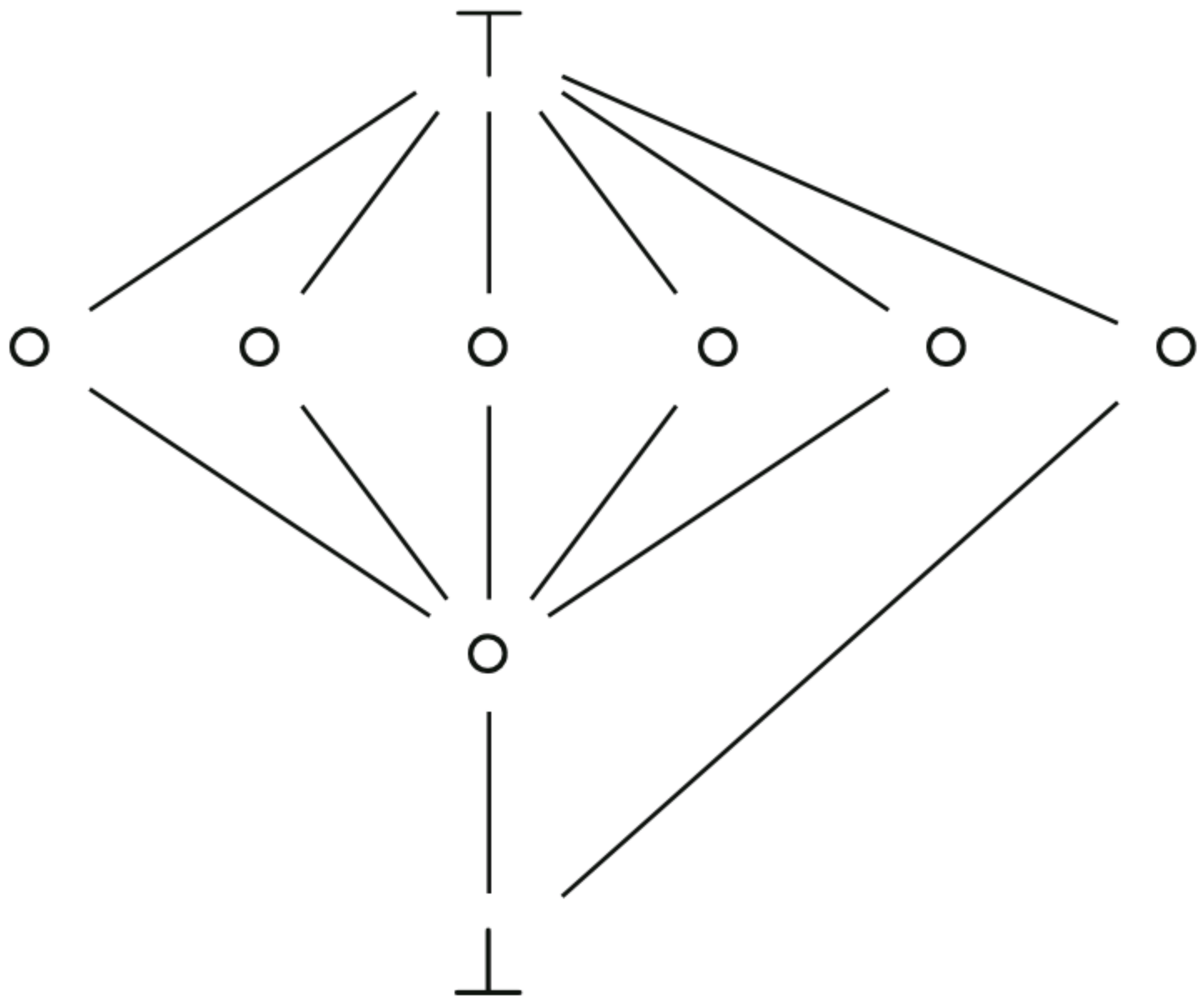




Results

Which lattice has the least quantales?

- Surprisingly, there is (only) one lattice on 9 elements with *even fewer* quantales than the diamond (19 447 vs 48 299):



Results

Miscellaneous insights from the data

- The choice of lattice seems to have influence on what properties a quantale can have.
- In particular, most lattices *do not* have integral / Frobenius quantales
- Some lattices have only one (trivial) two-sided quantal

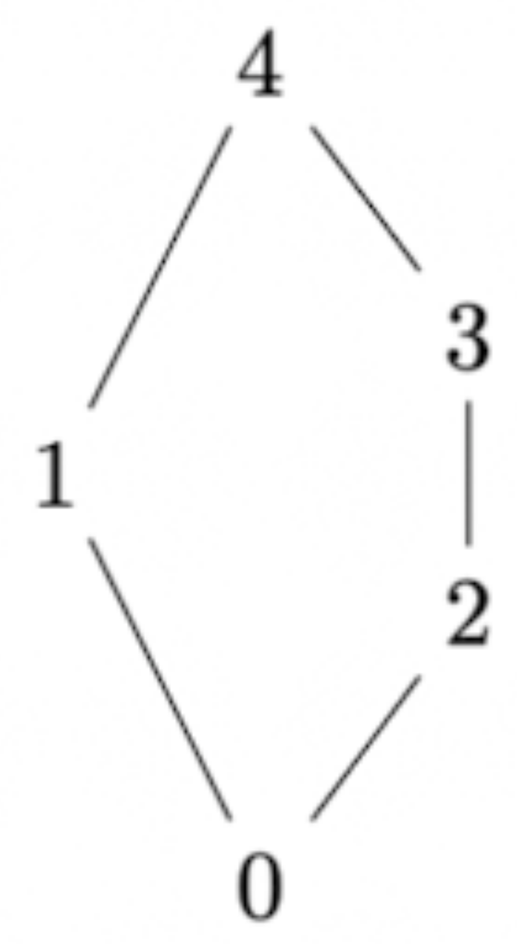
N	1	2	3	4	5	6	7	8	9
Lattices	1	1	1	2	5	15	53	222	1 078
With Integral	1	1	1	2	3	7	18	62	244
With Frobenius	1	1	1	2	3	7	13	36	76
With exactly 1 2-sided quant.	1	0	0	0	1	2	7	34	184

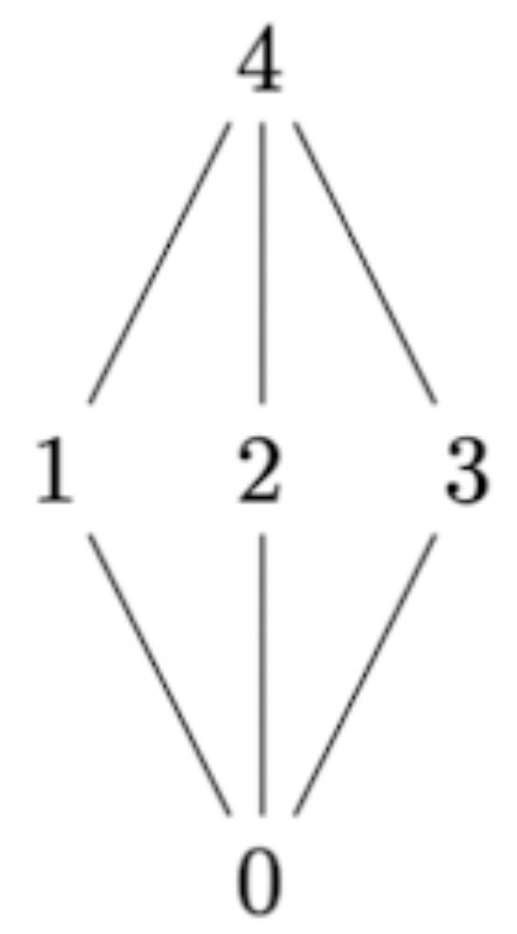
Results

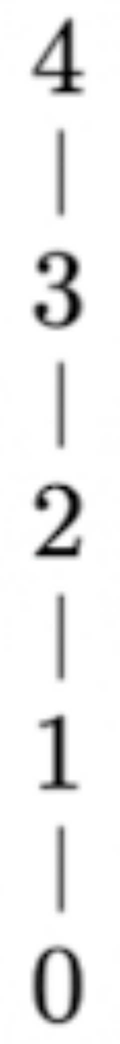
Which lattices can be part of Frobenius quantales?

5 elements

YES







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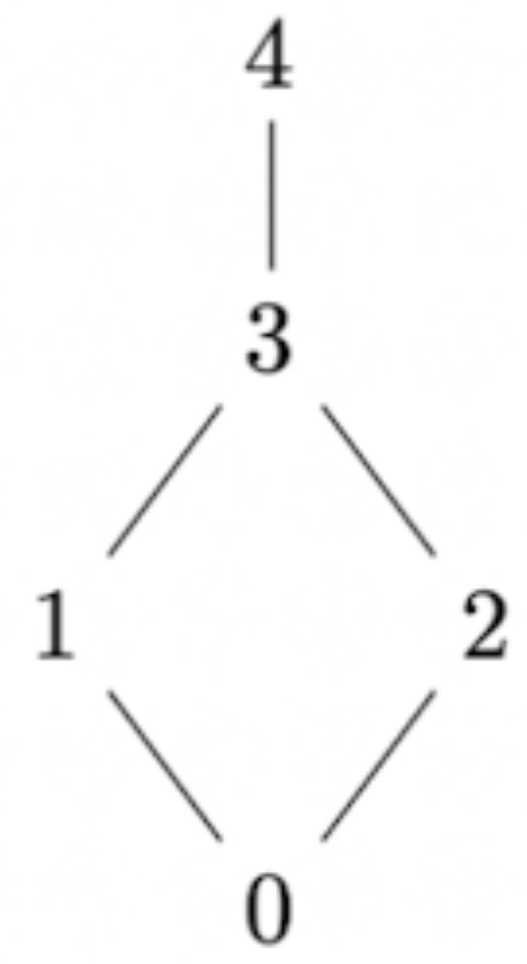
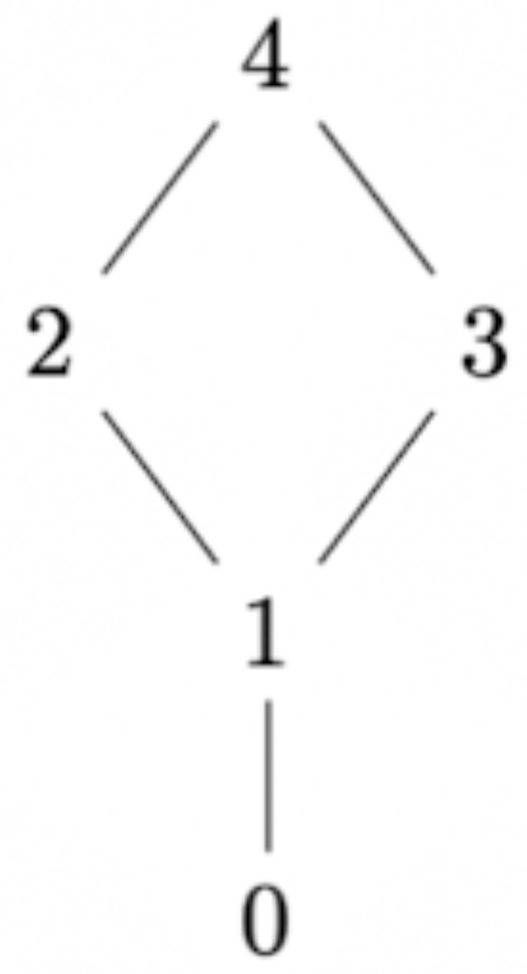


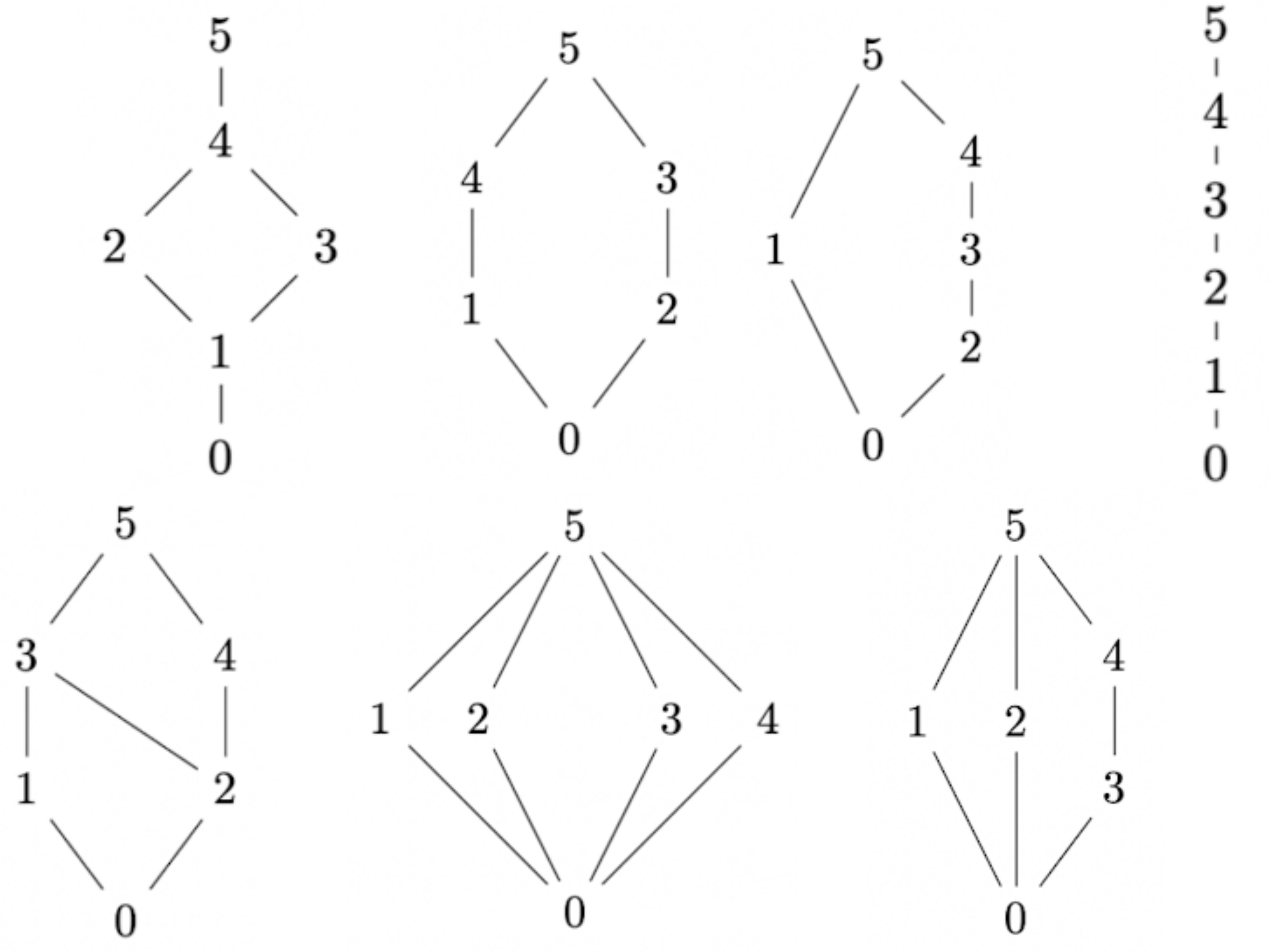


Results

Which lattices can be part of Frobenius quantales?

6 elements

YES



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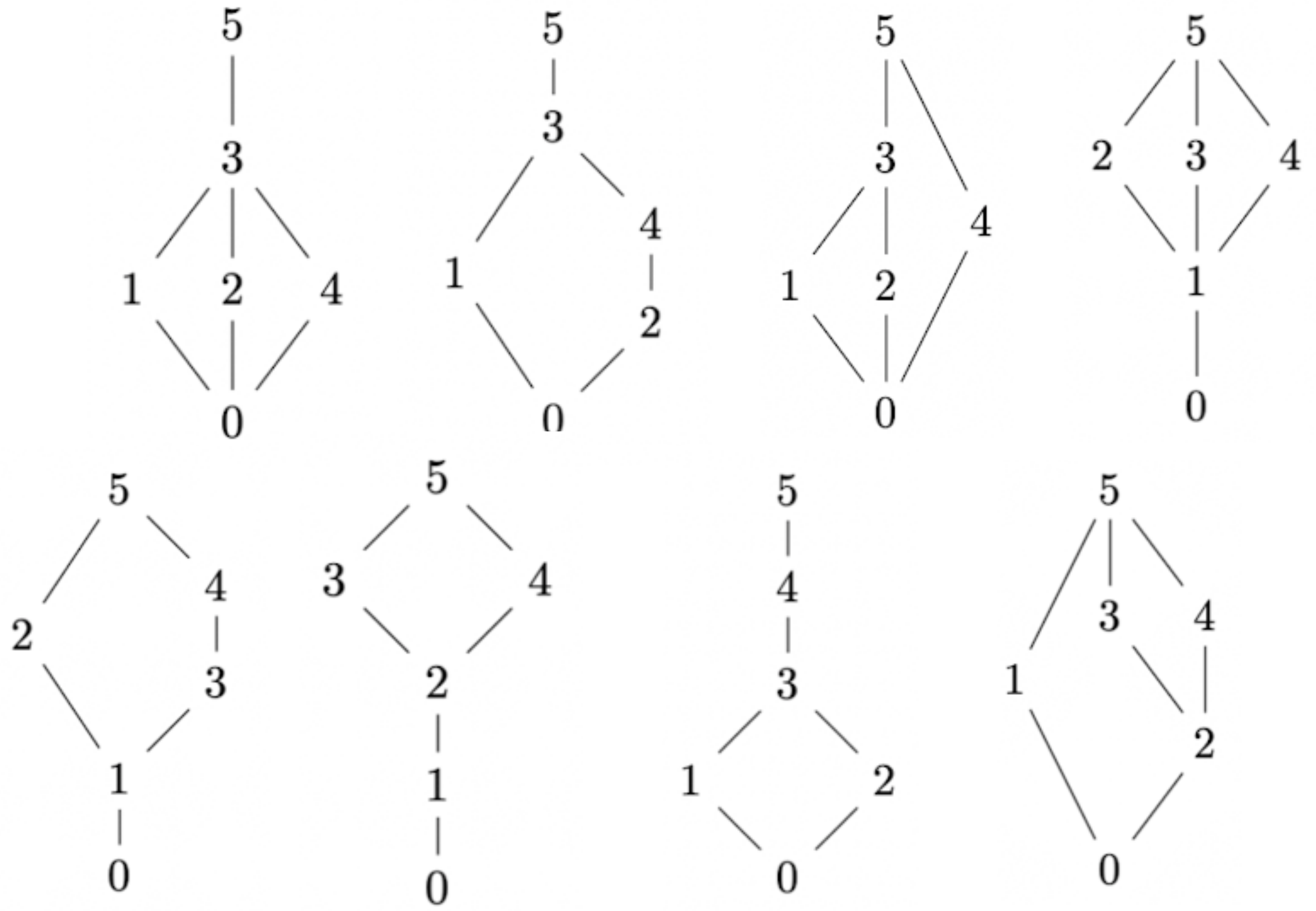
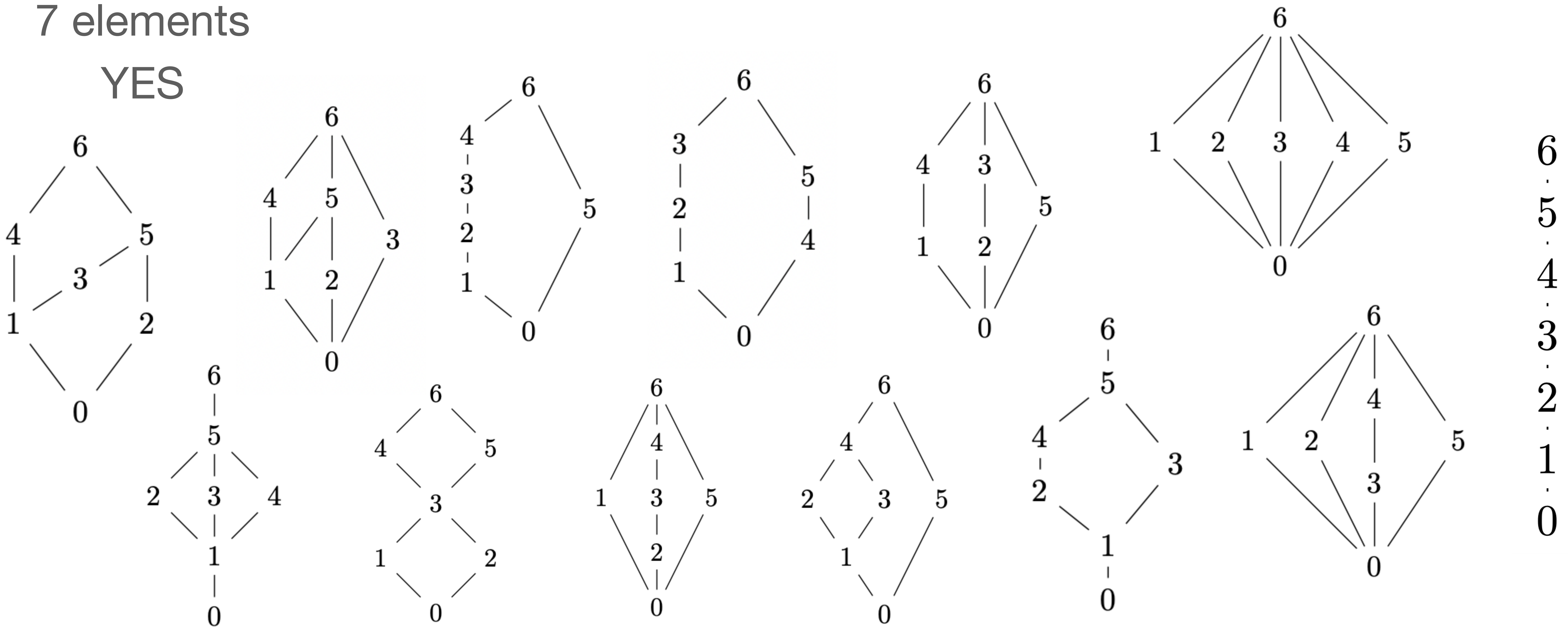


Results

Which lattices can be part of Frobenius quantales?

7 elements

YES



Results

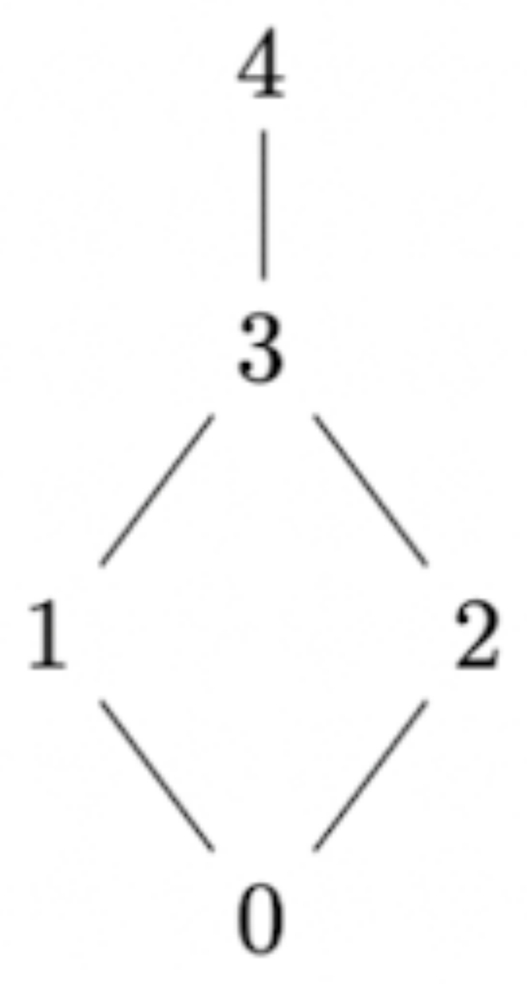
- Thm: Exactly the self-dual lattices admit Frobenius quantales (Known result…)
- Manually verified by drawing the corresponding Hasse diagrams for ≤ 8 elements and checking for self-duality.
- General proof: $x \mapsto x \to d$ and $x \mapsto d \leftarrow x$ are isomorphisms between a quantal and its opposite for dualizing elements d.

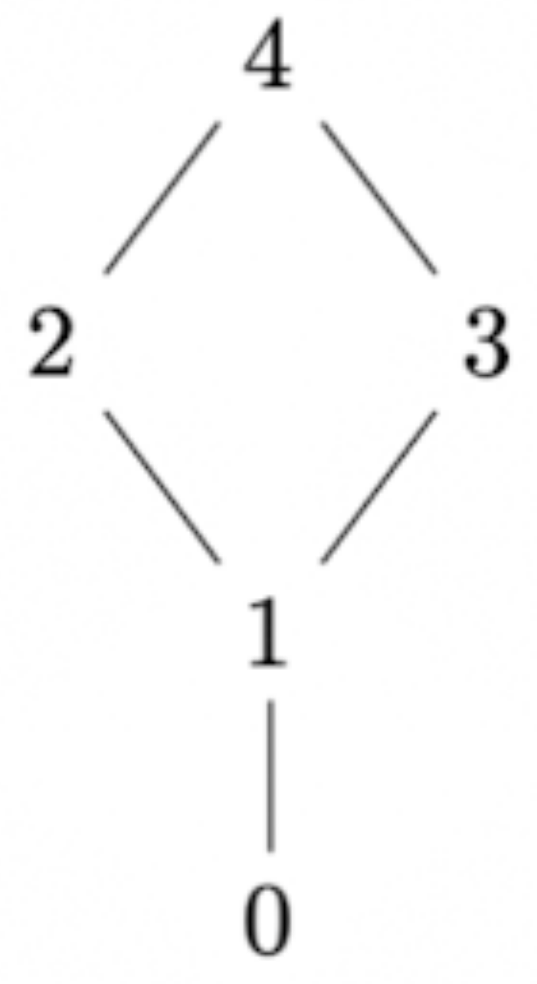
Results

Which lattices can be part of integral quantales?

5 elements

YES





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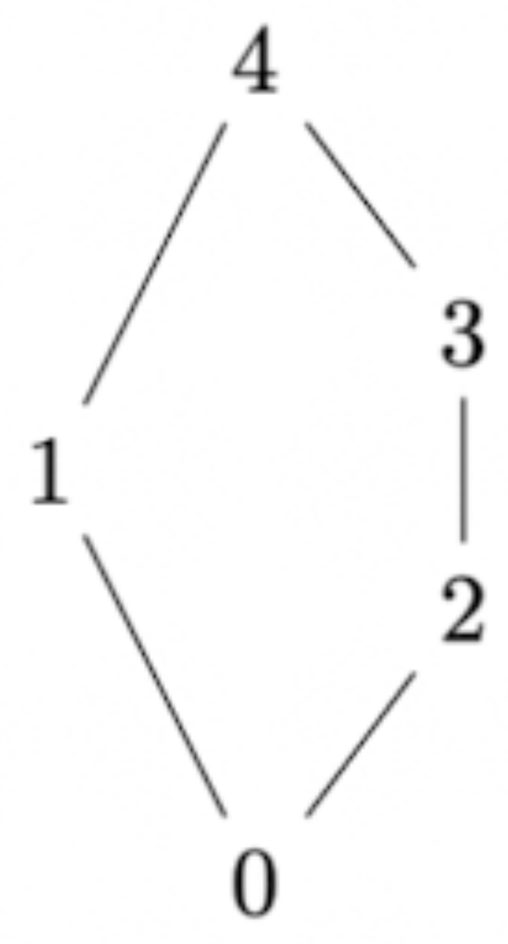
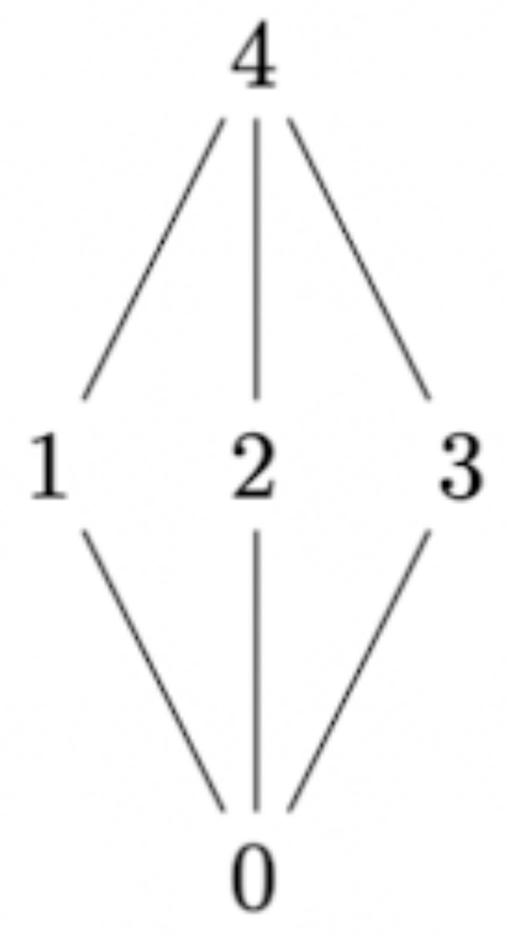


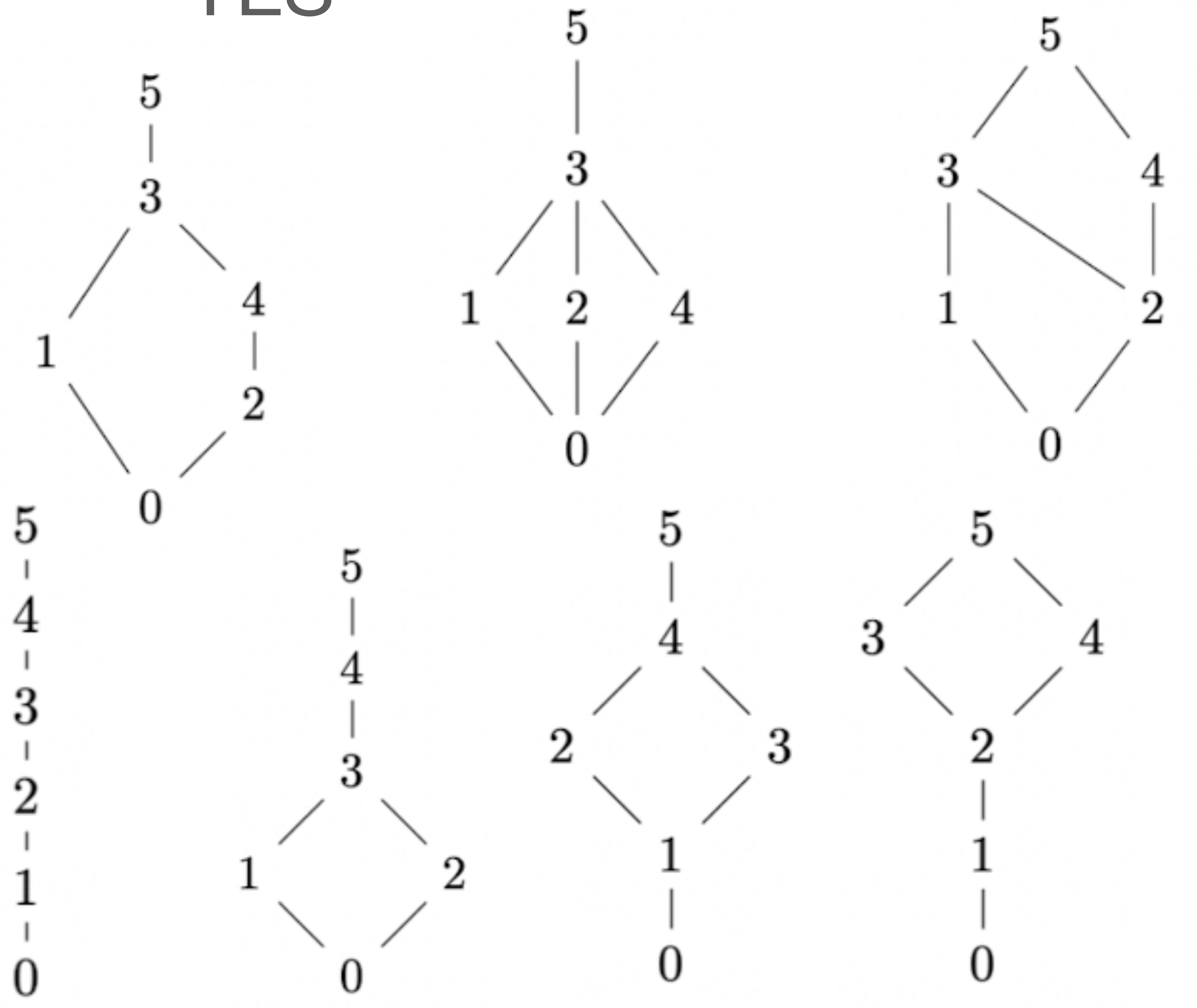


Results

Which lattices can be part of integral quantales?

6 elements

YES



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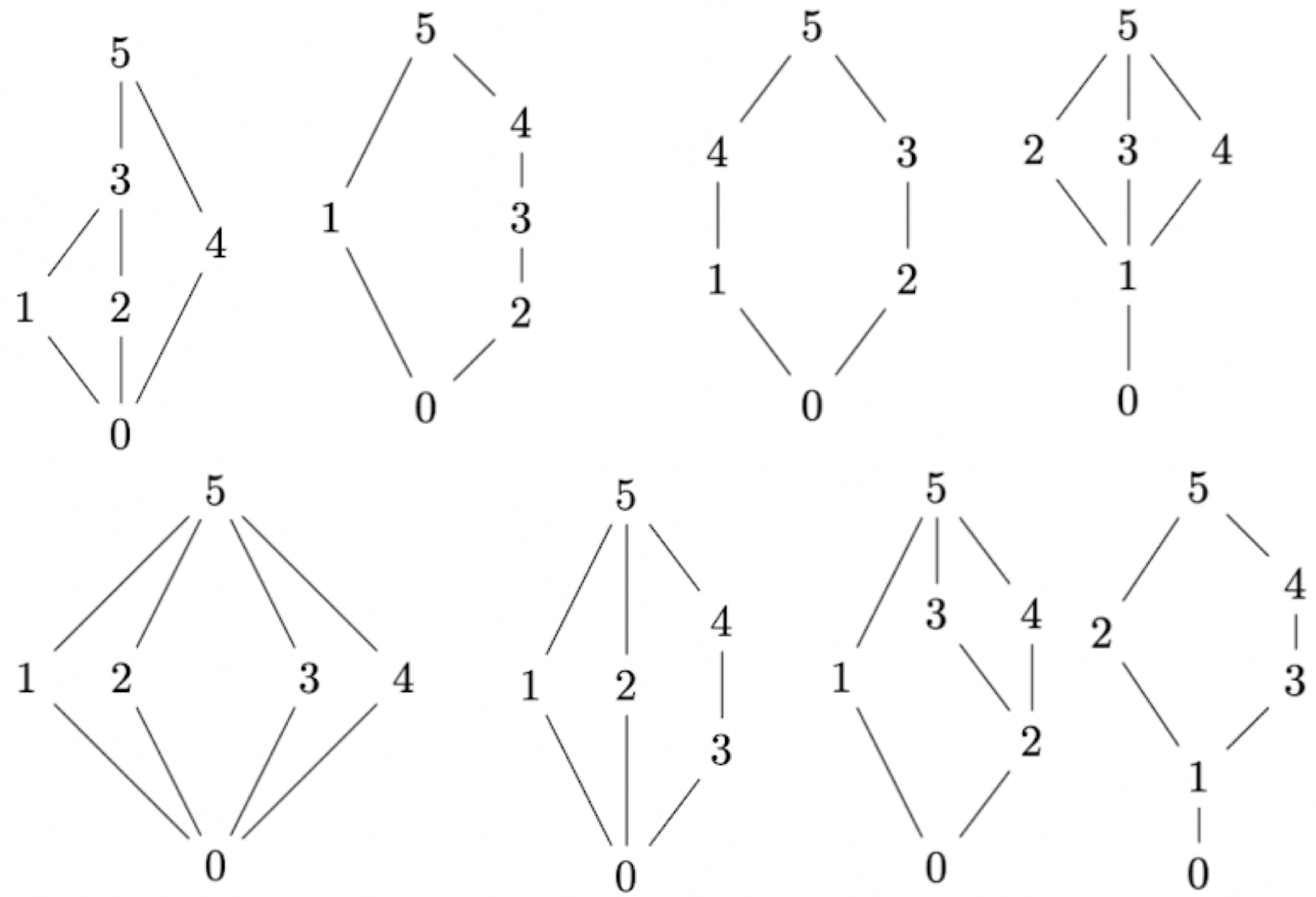
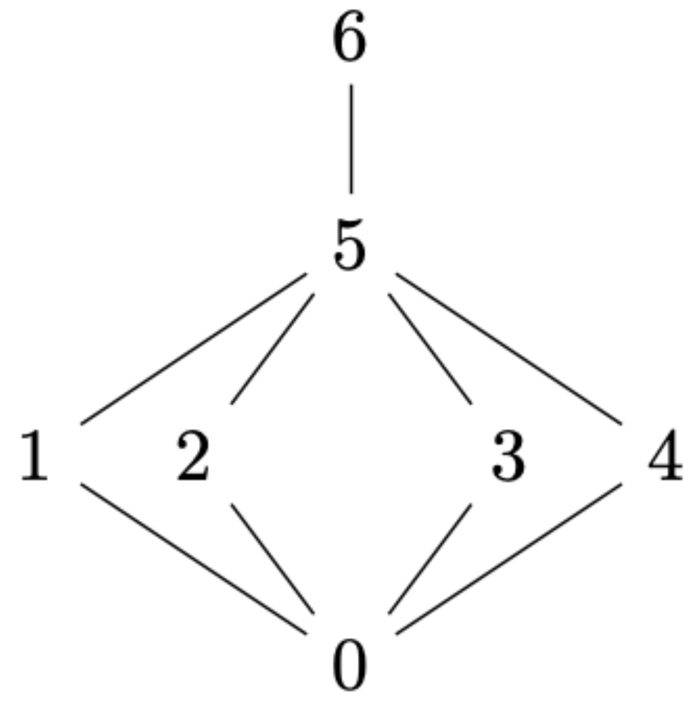


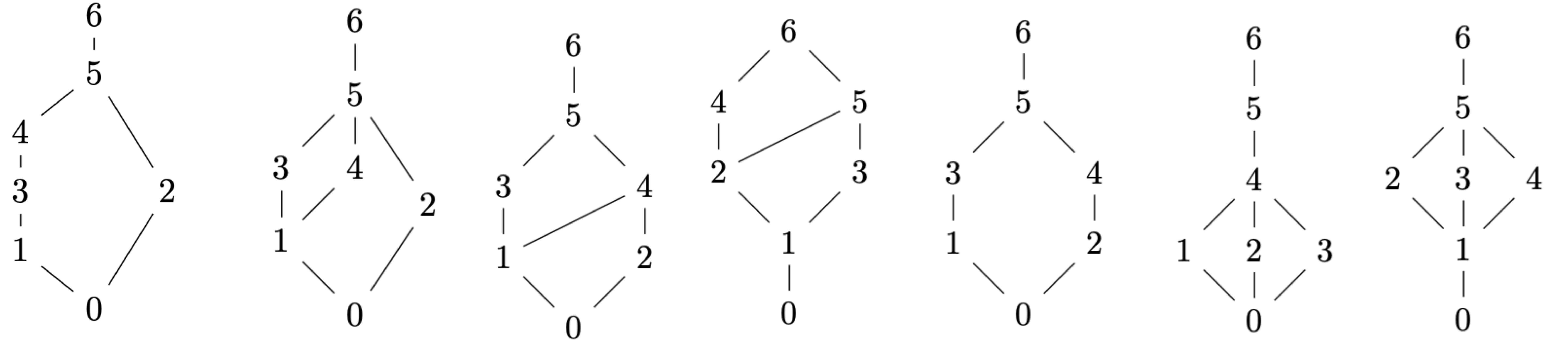
Results

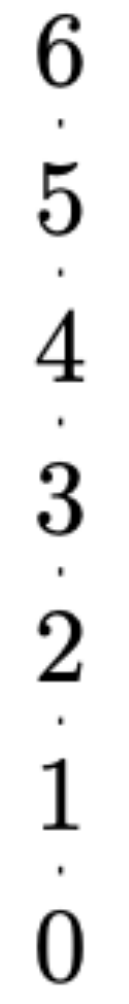
Which lattices can be part of integral quantales?

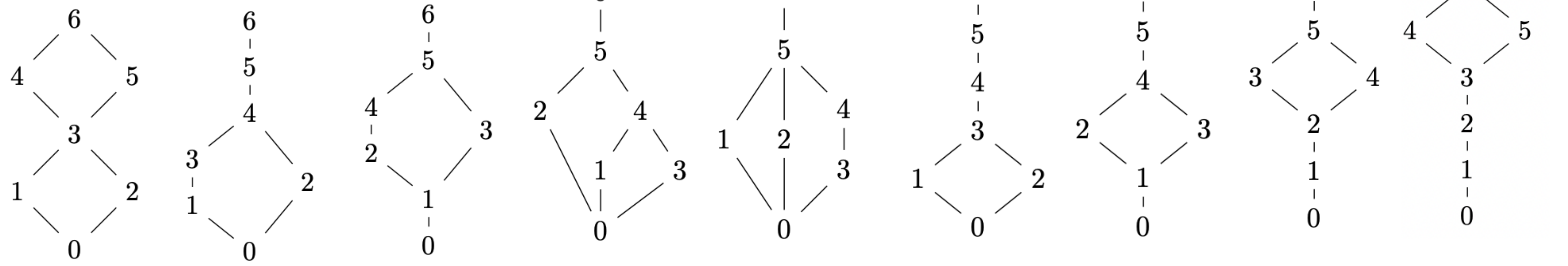
7 elements

YES









Results

- Open Problem: Categorize the lattices that admit an integral quantale
- Progress: A sufficient but not necessary condition for a lattice to not have integral quantales - to be published(?)
- Generalizes to a condition for specific elements of a lattice that cannot be units in any associated quantale

Results

- Open Problem: Categorize the lattices that admit only the trivial two-sided quantale
- Progress: A condition that is **both** sufficient and necessary for 9 elements and less - to be published(?)
- Several further questions and problems are being investigated by the author
- Broader Research Question: How does the choice of lattice influence the properties a quantale can have?

Results

Minimal quantales that are Frobenius, but not Girard

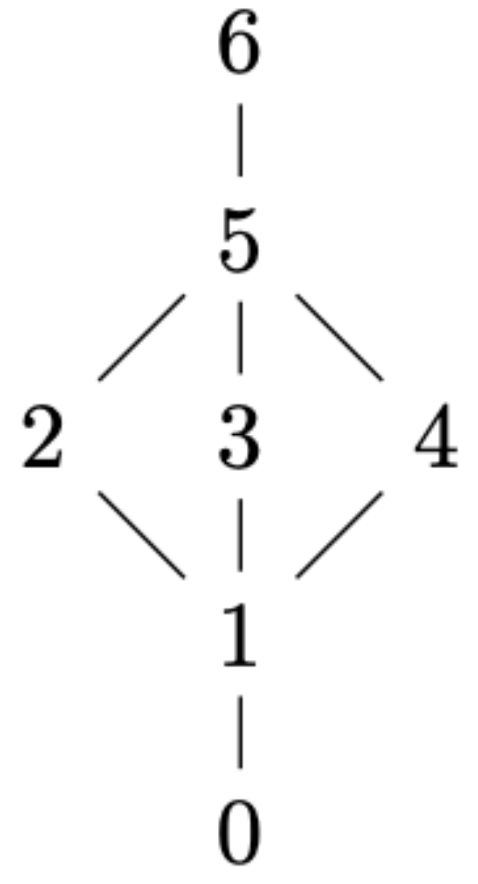
- Exercise 2.6.5e in Eklund et al: show that this quantale is Frobenius, but not Girard
- We can now conclude this is a *minimal example*

Lattice nr. 39, Quantale nr. 1540

Cyclic elements: 1,2,3,4,5,6. Dualizing element: 0

$*$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1
2	0	0	1	0	1	1	2
3	0	0	1	1	0	1	3
4	0	0	0	1	1	1	4
5	0	0	1	1	1	1	5
6	0	1	2	3	4	5	6

$\vee$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	2	3	4	5	6
2	2	2	2	5	5	5	6
3	3	3	5	3	5	5	6
4	4	4	5	5	4	5	6
5	5	5	5	5	5	5	6
6	6	6	6	6	6	6	6



Results

Minimal quantales that are Frobenius, but not Girard

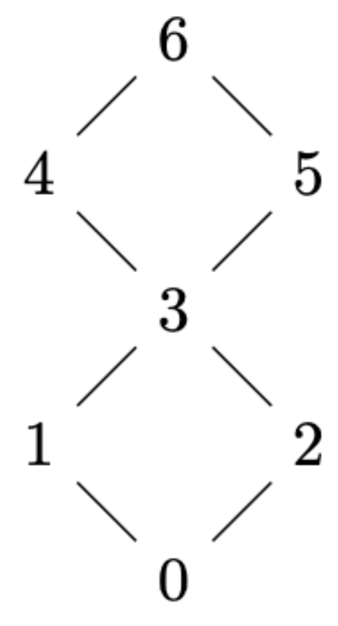
- In fact, there are 4 additional minimal examples, for a total of 5

Lattice nr. 25, Quantale nr. 2210

Cyclic elements: 3,4,5,6. Dualizing element: 0

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0	0	0	1	0	1
2	0	0	0	0	0	2	2
3	0	0	0	0	1	2	3
4	0	0	2	2	4	2	4
5	0	1	0	1	1	5	5
6	0	1	2	3	4	5	6

∨	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	3	3	4	5	6
2	2	3	2	3	4	5	6
3	3	3	3	3	4	5	6
4	4	4	4	4	4	6	6
5	5	5	5	5	6	5	6
6	6	6	6	6	6	6	6

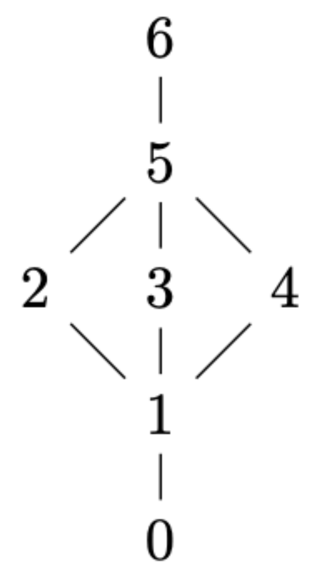


Lattice nr. 39, Quantale nr. 3299

Cyclic elements: 0,1,2,3,5,6. Dualizing element: 4

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	6	6	5	6	6
3	0	3	5	6	6	6	6
4	0	4	6	5	6	6	6
5	0	5	6	6	6	6	6
6	0	6	6	6	6	6	6

∨	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	2	3	4	5	6
2	2	2	2	5	5	5	6
3	3	3	5	3	5	5	6
4	4	4	5	5	4	5	6
5	5	5	5	5	5	5	6
6	6	6	6	6	6	6	6

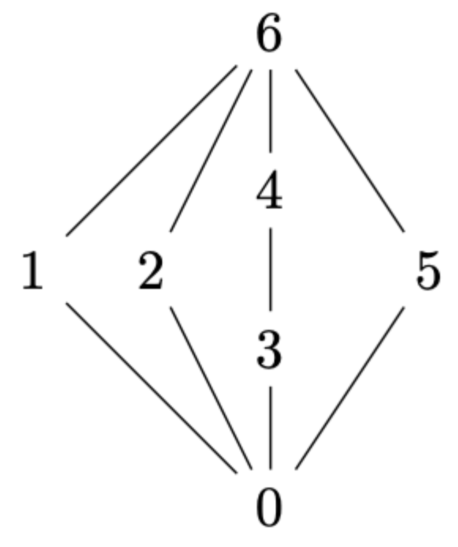


Lattice nr. 28, Quantale nr. 7772

Cyclic elements: 0,1,2,3,4,6. Dualizing element: 5

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	6	6	1	6	4	6
2	0	4	6	2	6	6	6
3	0	1	2	3	4	5	6
4	0	6	6	4	6	6	6
5	0	6	4	5	6	6	6
6	0	6	6	6	6	6	6

∨	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	6	6	6	6	6
2	2	6	2	6	6	6	6
3	3	6	6	3	4	6	6
4	4	6	6	4	4	6	6
5	5	6	6	6	6	5	6
6	6	6	6	6	6	6	6

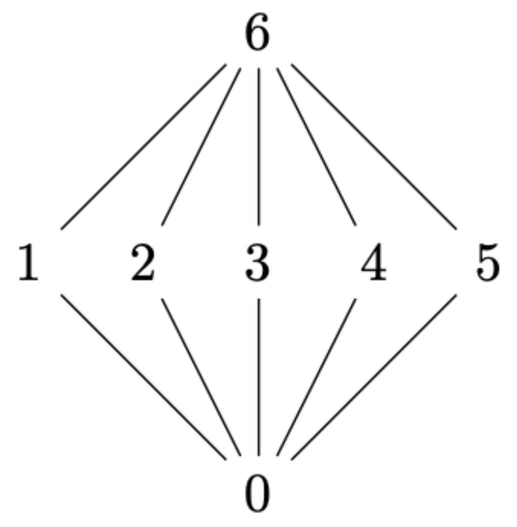


Lattice nr. 27, Quantale nr. 154

Cyclic elements: 0,1,2,3,4,6. Dualizing element: 5

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	6	6	5	6	6
3	0	3	5	6	6	6	6
4	0	4	6	5	6	6	6
5	0	5	6	6	6	6	6
6	0	6	6	6	6	6	6

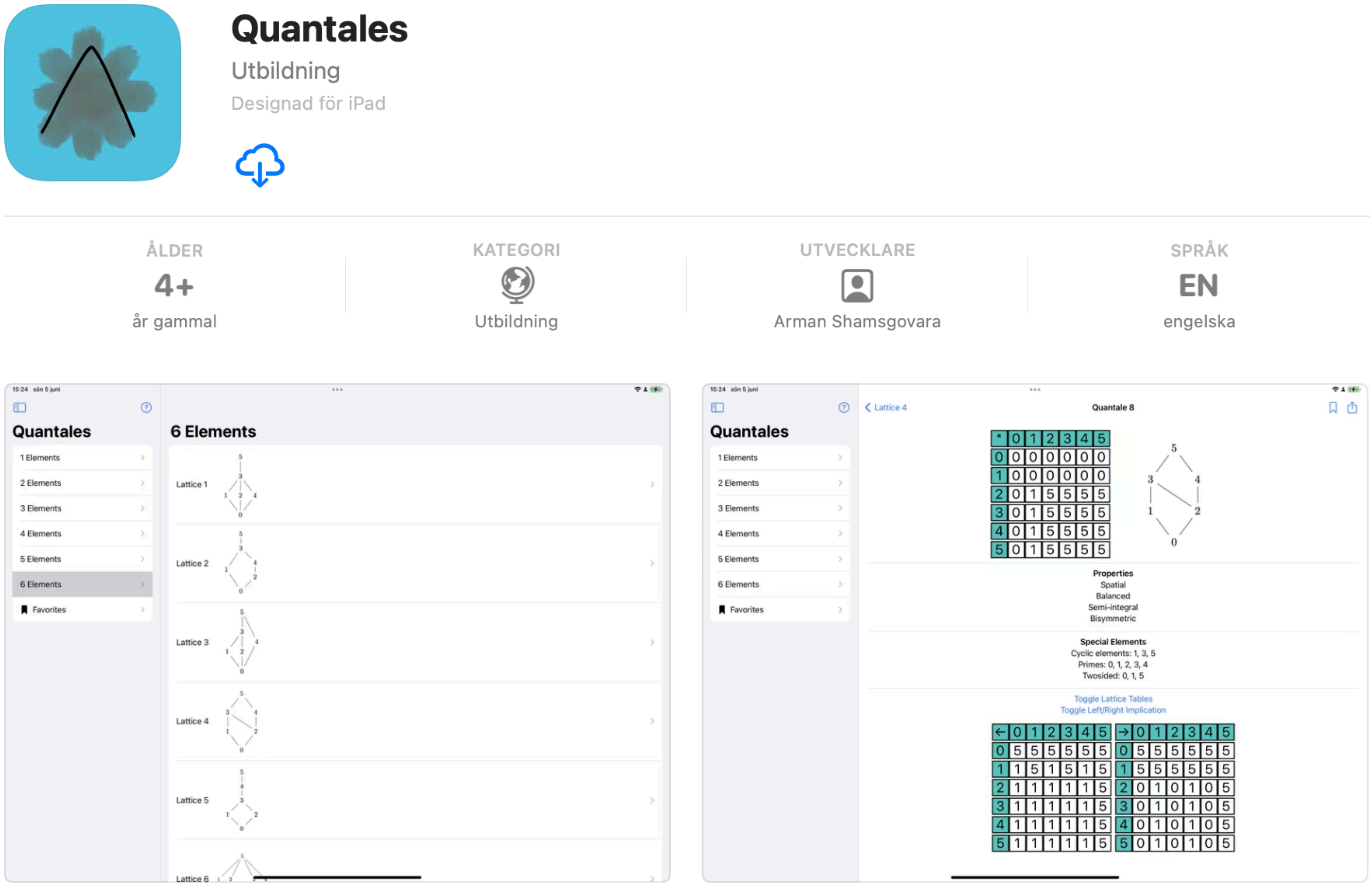
∨	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	6	6	6	6	6
2	2	6	2	6	6	6	6
3	3	6	6	3	6	6	6
4	4	6	6	6	4	6	6
5	5	6	6	6	6	5	6
6	6	6	6	6	6	6	6



Bonus

iOS / Mac App

- Basic app for browsing quantales
- Up to 6 elements supported
- Filtering options
- Favorites / Sharing
- ≈ 20 MB in size!
- Android version TBA



Future Work

4 Main Avenues

Enumeration

- 10 elements
- Same techniques, other structures (semirings…)
- More efficient branching schemes / symmetry breaking

Analysis

- Chart out the space of finite quantales
- Patterns, numbers, examples

Applications

- Diagnosis systems
- Finite quantales in logic?

Availability

- Android app / web service
- GAP Package?
- OEIS entries (underway)

Thank you!