Linear bounds between Cliquewidth and Component twin-width and applications

Ambroise Baril, Miguel Couceiro, Victor Lagerkvist

Université de Lorraine, CNRS, LORIA & Linköpings Universitet

April 2023
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1. **#H-COLORING**

2. First bound

3. Second bound

4. Complexity consequences
$k$-COLORING

Figure: Instance of 3-COLORING

Figure: Solution of the instance

$c : V_G \rightarrow [k]$ such that $\forall (u, v) \in E_G, c(u) \neq c(v)$
Example of a $C_5$-COLORING

$f : V_G \rightarrow V_H$

$\forall (u, v) \in E_G, (f(u), f(v)) \in E_H$

$f$ is an Homomorphism

$k$-COLORING $= K_k$-COLORING
Hard problem

Naive algorithm in time $|V_H|^{|V_G|}$

No algo in time $F(H) \times |V_G|^{F(H)}$ unless $P=NP$ ($H = K_3$)

No algo in time $F(G) \times |V_H|^{O(1)}$ unless $FPT=W[1]$ ($G = K_k$)

How to solve in practice?
Hard problem

Naive algorithm in time $|V_H|^{|V_G|}$

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How to solve in practice?

Use structural properties of the graphs involved
Clique-width

\( \bullet_i \): vertex labelled by \( i \)

\( G_1 \oplus G_2 \): disjointed union

\( \rho_{j \rightarrow i}(G) \): relabel the \( j \) with \( i \)

\( \eta_{i,j}(G) \): construct an edge between every \( i \) and \( j \)

\( \text{cw}(G) \): number of labels

\( \text{linearcw}(G) \): number of labels where every \( \oplus \) contains a \( \bullet_i \) member
Application to counting homomorphisms

Parameterized complexity:

\[ k\text{-COLORING in time } (2^{|V_H|} - 2)^{cw(G)} \]  [Lam20]\(^1\)

Fine-grained complexity:

\[ \#H\text{-COLORING in time:} \]
\[ (2^{cw(H)} + 1)^{|V_G|} \text{ and } (\text{linear}cw(H) + 2)^{|V_G|} \]  [Wah11]\(^2\)

\(^1\)Lampis
\(^2\)Wahlström
Exemple of a contraction sequence

Figure: A contraction sequence of a graph
(Component) twin-width

**Figure:** Contraction sequence of a graph

$tww(G)$: Maximal red-degree [BKTW20]$^a$

$ctww(G)$: Max red-component size [BKRT22]$^b$

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$^a$Bonnet, Kim, Thomassé, Watrigant

$^b$Bonnet, Kim, Reinald, Thomassé

No FPT algo for 3-COLOR param by $tww(G)$:

3-COLOR is NP-hard on planar graphs

tww is bounded on planar graphs
Applications to counting homomorphisms

Naive use of component twin-width for \(\#H\)-COLORING:

Parameterized complexity:

\[(2^{|V_H|} - 1)^{ctww(G)}\]

Fine-grained complexity:

\[(ctww(H) + 2)^{|V_G|}\]
Comparing complexities

Which approach is the best?

Parameterized complexity:

\[(2^{V_H} - 1)^{ctww(G)} \text{ VS } (2^{V_H} - 2)^{cw(G)}\]

Fine-grained complexity:

\[(ctww(H) + 2)^{|V_G|} \text{ VS } (2cw(H) + 1)^{|V_G|} \text{ and } (linearcw(H) + 2)^{|V_G|}\]

We need to compare the two parameters \(cw\) and \(ctww\).
Using boolean-width (func equiv to cliquewidth) [BKRT22]\(^3\)

\[
\text{ctww}(G) \leq 2^{\text{boolw}(G)} + 1 \leq 2^{\text{cw}(G)} + 1
\]

AND

\[
\text{cw}(G) \leq 2^{\text{boolw}(G)} \text{ and boolw}(G) \leq 2^{\text{ctww}(G)}
\]

so

\[
\text{cw}(G) \leq 2^{2^{\text{ctww}(G)}}
\]

\(^3\)Bonnet, Kim, Reinald, Thomassé
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We already know:

$$cw(G) \leq 2^{ctww(G)}$$
First contribution: Improved bound

I will prove

\[ \text{cw}(G) \leq \text{ctww}(G) + 1 \]

Take a contraction sequence of \( G \) of \( \text{ctww} \) \( k \)

Build a \((k+1)\)-expression of \( G \)
Exemple of a contraction sequence

For \( C = \{S_1, \ldots, S_p\} \) red-component
Build \( \varphi_C \) a \((k + 1)\)-expression of \( G[S_1 \uplus \cdots \uplus S_p] \) with \( \forall i, \text{label}(S_i) = i \)

Same red-component = Same formula
Same set = Same label
Contraction sequence of ctww=3

We will use 4 labels: •, •, •, •: proves cw ≤ 4

Red-component are singletons \{a\}, \{b\}, …
Contracting e and f

\[ \varphi_a = \bullet \]
\[ \varphi_b = \bullet \]
\[ \varphi_c = \bullet \]
\[ \varphi_d = \bullet \]
\[ \varphi_e = \circ \]
\[ \varphi_f = \bullet \]
\[ \varphi_g = \bullet \]

\[ \varphi_{a \text{def}} = \]
\[ \rho \bullet \rightarrow \bullet \]
\[ \eta \circ, \eta \circ, \eta \circ, \bullet \]
\[ (\varphi_a \oplus \varphi_d \oplus \varphi_e \oplus \varphi_f) \]
Contracting $a$ and $d$

$\varphi_{abcdef} = \bullet$

$\varphi_g = \bullet$

$\varphi_{abcdefg} = \rho \bullet \cdots \bullet \eta \bullet \cdots \eta \cdots$

$(\varphi_{abcdef} \oplus \varphi_g)$
Contracting $b$ and $ef$

\[ \varphi_{\text{adef}g} \]
\[ \varphi_b \]

\[ \varphi_{\text{adbef}g} = \rho \oplus \eta_{\bullet, \eta_{\bullet, \bullet}} \]
\[ (\varphi_{\text{adef}g} \oplus \varphi_b) \]
Contracting \( ad \) and \( g \)

\[
\varphi_{adbefg} = \rho \rightarrow \varphi_{adgbefg}
\]
Contracting $c$ and $bef$

$$\varphi_{adg bef} = \rho \cdot \rightarrow \cdot \eta \cdot \cdot \cdot \left( \varphi_{adg bef} \oplus \varphi_c \right)$$
Consequence

Contraction of comp.width $k \implies (k + 1)$-expression

$$cw(G) \leq ctww(G) + 1$$

Tight for cographs ($cw = 2$, $ctww = 1$)
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We already know:

$$\text{ctww}(G) \leq 2^{\text{ctww}(G)} + 1$$
Second contribution: Improved bound on component twin-width

I will prove

$$\text{ctww}(G) \leq 2\text{cw}(G) - 1 \text{ and } \text{ctww}(G) \leq \text{linearcw}(G)$$

Take a (linear) $k$-expression

Build a contraction sequence of $G$, where every red-component has size $\leq 2k - 1$ (resp. $\leq k$).
$k$-expression

Figure: $k$-expression tree structure

Severe abuse of notation: $\oplus$ must be binary
Intuition: contract same colors in $\oplus$

Build larger and larger "parks" following the $k$-expressions.

Contract similar colors:
- Parks size $\leq 2k$
- No red-edges crossing parks

Initial parks are single vertices
Free contraction of twins

Here, \(d, e\) and \(f\) (as well as \(h\) and \(i\)) are introduced together with the same labels: they are twins.
Contracting similar colors in a park

- Merge the parks of $a$ and $b$, of $c$ and $def$ and of $g$ and $hi$.
- Collapse the $k$-expression.
- No 2 different colors in the same park: no contraction.

$\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$

$\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$

$\rho \cdot \cdot \cdot$ $\oplus$ $\rho \cdot \cdot \cdot$ $\oplus$ $\rho \cdot \cdot \cdot$ $\oplus$ $\rho \cdot \cdot \cdot$ $\oplus$

$\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$

$\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$ $\eta \cdot \cdot$ $\oplus$
Joining different colors in a park

- Merge the parks of \{a, b\} and \{c, def\} and of \{g, hi\} and \{j\}.
- \(b\) and \(c\) are both blue in the same park: contract them.
Main argument: no red-edge crossing parks

\[
\begin{align*}
abc & \text{def} \\
\rho & \rightarrow \\
\eta & , , \\
ghij & \\
\rho & \rightarrow \\
\end{align*}
\]

\[
\begin{array}{cccc}
b & c & \text{will have eternally the same label} \\
b & c & \text{have exactly the same neighbors in} \{g, h, i, j\}: \text{no red-edge crossing parks} \\
\end{array}
\]

\[
\begin{align*}
b & \text{and} \ c \ \text{have been contracted.} \\
b & \text{and} \ c & \text{will become blue: contract} \ a \\
j & \text{will become green: contract} \ j & \text{and} \ g
\end{align*}
\]
Renaming in a park: no red-edge crossing parks

\[ \eta, \cdot \]

\( g \) and \( j \) will have eternally the same label.

\( g \) and \( j \) have exactly the same neighbors in \( \{a, b, c, d, e, f\} \)

Next step: merge parks.
One park left: Ends.
Finish the contraction sequence randomly.
Largest possible red-component

\[\eta, \eta\]

\[123\] \[1'2'3'\]

\[\oplus\]

\(k\) labels on both side.
Red-comp of size \(k\) on both side.

Peak: Red-comp of size \(2k - 1\)
Then, contract by color until \(k\) vertices left in the park
Then, proceed to the next \(\oplus\)
Case of a linear $k$-expression

**Linear $k$-expression:** $G_1 \oplus G_2$ is used $\implies$ $G_2$ has one vertex

- $123$ and $2'$
- $\eta\cdot\cdot$

$k$ labels on one side.
1 vertex (so 1 label) on the otherside

Peak: Red-comp of size $k$
Consequence

(Linear) $k$-expression $\implies$ contraction sequence with every red-comp having size $\leq 2k - 1$ (resp. $k$)

$$\text{ctww}(G) \leq 2\text{cw}(G) - 1 \quad \text{and} \quad \text{ctww}(G) \leq \text{linearcw}(G)$$

$$\text{tww}(G) \leq 2\text{cw}(G) - 2$$

Tight ?
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Parameterized complexity

Use the first bound: \( cw \leq ctww + 1 \)

\[(2^{|V_H| - 2})^{cw(G)} \text{ VS } (2^{|V_H| - 1})^{ctww(G)}\]
Parameterized complexity

Use the first bound: \(cw \leq ctww + 1\)

\[(2^{|V_H|} - 2)^{cw(G)} \text{ VS } (2^{|V_H|} - 1)^{ctww(G)}\]

Clique-width approach wins... for the moment (very naive)!
Fine grained complexity

Use the second bound: \( \text{ctww} \leq 2\text{cw} - 1 \) and \( \text{ctww} \leq \text{linearcw} \)

\[
(\text{ctww}(H) + 2)^{|V_G|} \quad \text{VS} \quad (2\text{cw}(H) + 1)^{|V_G|} \quad \text{and} \quad (\text{linearcw}(H) + 2)^{|V_G|}
\]
Use the second bound: \( \text{ctww} \leq 2\text{cw} - 1 \) and \( \text{ctww} \leq \text{linear}\text{cw} \)

\[(\text{ctww}(H) + 2)|V_G| \quad \text{vs} \quad (2\text{cw}(H) + 1)|V_G| \quad \text{and} \quad (\text{linear}\text{cw}(H) + 2)|V_G|\]

Component twin-width approach wins without effort
References


Thank you for your attention!

Questions?