EQUATIONAL THEORIES AND DISTANCES FOR COMPUTATIONAL EFFECTS

Valeria Vignudelli

CNRS, ENS Lyon

IN THIS TALK

Computational effect (monad in Set)	$\Leftrightarrow$	Equational Theory (Σ, E) for Σ a signature, E a set of equations

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Effects: nondeterminism, probabilities, termination, combinations

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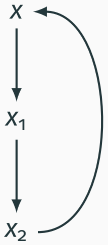

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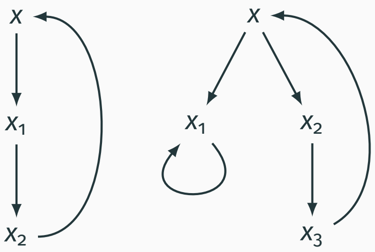

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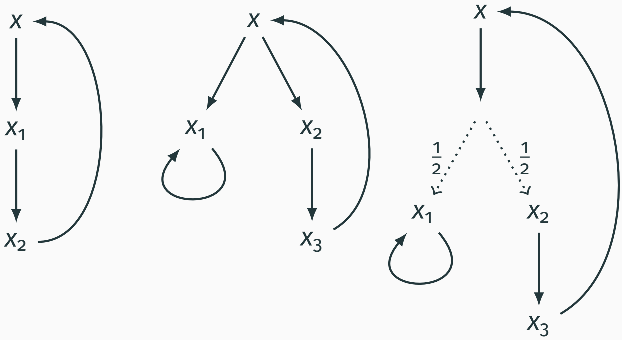

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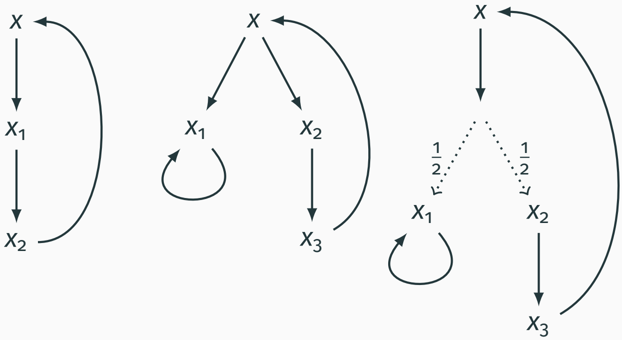


- reasoning equationally on equivalences of systems
- what about reasoning equationally on distances?

Computational effect (monad in Set) $\iff$ Equational Theory (Σ, E) for Σ a signature, E a set of equations

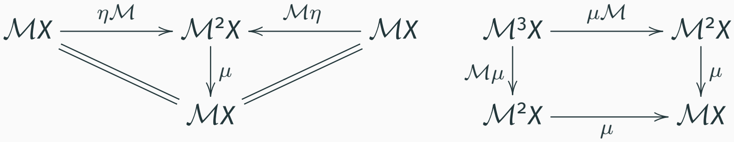
↓

Effect+distance (monad in Met) $\iff$ Quantitative Equational Theory (Σ, Q) for Σ a signature, Q quantitative inferences

Monads and Equational Theories for Computational Effects

Monad $(\mathcal{M}, \eta, \mu)$
in Set

- functor $\mathcal{M} : X \mapsto \mathcal{M}(X)$
- unit $\eta_X : X \to \mathcal{M}(X)$
- multiplication $\mu_X : \mathcal{M}\mathcal{M}(X) \to \mathcal{M}(X)$



Monad $(\mathcal{M}, \eta, \mu)$
in Set

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

- terms $t := x | op(t_1, ...t_n)$ for $op \in \Sigma$
- E a set of equations $\quad t = s$

Deductive system: equational logic

(Reflexivity) $\emptyset \vdash t = t$

(Symmetry) $\{t = s\} \vdash s = t$

(Transitivity) $\{t = u, u = s\} \vdash t = s$

Models: algebras (A, Σ^A) satisfying E

Free model: $(Terms(X, \Sigma)_{/E}, \Sigma)$

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

(Σ, E) is a presentation of $(\mathcal{M}, \eta, \mu)$

The category **EM**$(\mathcal{M})$ of Eilenberg-Moore algebras for $(\mathcal{M}, \eta, \mu)$ is isomorphic to the category **A**(Σ, E) of algebras (models) of (Σ, E)

Category **EM**$(\mathcal{M})$

- objects: $(A, \alpha : \mathcal{M}(A) \to A)$ with α commuting with η, μ
- arrows: algebra morphisms

Category **A**(Σ, E)

- objects: models (A, Σ^A) of (Σ, E)
- arrows: homomorphisms of (Σ, E)-algebras

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\iff$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

(Σ, E) is a presentation of $(\mathcal{M}, \eta, \mu)$

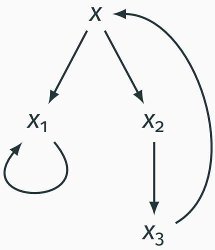
The category $\mathbf{EM}(\mathcal{M})$ of Eilenberg-Moore algebras for $(\mathcal{M}, \eta, \mu)$ is isomorphic to the category $\mathbf{A}(\Sigma, E)$ of algebras (models) of (Σ, E)

Corollary: equational reasoning on free objects

Free algebra for the monad $\cong (Terms(X, \Sigma)_{/E}, \Sigma)$

EXAMPLE: NONDETERMINISM

Monad $(\mathcal{M}, \eta, \mu)$ in Set	$\Longleftrightarrow$	Equational Theory (Σ, E) for Σ a signature, E a set of equations



$\tau : X \to \mathcal{P}(X)$

$\tau(x) = \{x_1, x_2\}$
$\tau(x_1) = \{x_1\}$
...

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Powerset (non-empty) monad $(\mathcal{P}, \eta, \mu)$

- $\mathcal{P} : X \mapsto \{S \mid S \text{ is a non-empty, finite subset of } X\}$
- $\eta : x \mapsto \{x\}$
- $\mu : \{S_1, \ldots, S_n\} \mapsto \bigcup_i S_i$

$\Longleftrightarrow$

Equational theory of semilattices

- Σ = binary operation $\oplus$
- axioms of E =

$$(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)$$
$$x \oplus y \overset{(C)}{=} y \oplus x$$
$$x \oplus x \overset{(I)}{=} x$$

Monad $(\mathcal{M}, \eta, \mu)$
in Set

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Equational Theory (Σ, E)
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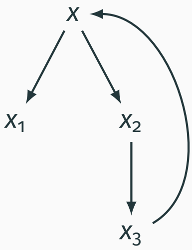
$$(\mathcal{P}(X), \bigcup) \cong (Terms(X, \Sigma)_{/E}, \oplus)$$

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations



$\tau : X \to \mathcal{P}^{\emptyset}(X)$

$\tau(x) = \{x_1, x_2\}$
$\tau(x_1) = \emptyset$
...

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Powerset (possibly empty) monad $(\mathcal{P}^{\emptyset}, \eta, \mu)$

- $\mathcal{P}^{\emptyset} : X \mapsto \{S \mid S$ is a finite subset of $X\}$
- $\eta : x \mapsto \{x\}$
- $\mu : \{S_1, \ldots, S_n\} \mapsto \bigcup_i S_i$

$\Longleftrightarrow$

Equational theory of semilattices with bottom

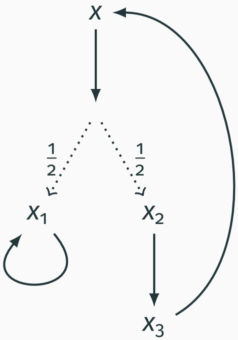
- $\Sigma = \star, \oplus$
- axioms of E=
  - axioms of semilattices

$$(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)$$
$$x \oplus y \overset{(C)}{=} y \oplus x$$
$$x \oplus x \overset{(I)}{=} x$$

  - bottom axiom $\quad x \oplus \star = x$

$$(\mathcal{P}^{\emptyset}(X), \bigcup, \emptyset) \cong (Terms(X, \Sigma)_{/E}, \oplus, \star)$$

Monad $(\mathcal{M}, \eta, \mu)$ in Set $\iff$ Equational Theory (Σ, E) for Σ a signature, E a set of equations



$\tau : X \to \mathcal{D}(X)$

$\tau(x) = \frac{1}{2}\, x_1 + \frac{1}{2}\, x_2$
$\tau(x_1) = 1\, x_1$
...

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Distribution monad $(\mathcal{D}, \eta, \mu)$

- $\mathcal{D} : X \mapsto \{\Delta \mid \Delta$ is a finitely supported probability distribution on $X\}$
- $\eta : x \mapsto 1x$
- $\mu : \sum_i p_i \Delta_i \mapsto \sum_i p_i \cdot \Delta_i$

$\Longleftrightarrow$

Equational theory of convex algebras

- Σ = binary operations $+_p$ for all $p \in (0, 1)$
- axioms of E =

$$(x +_q y) +_p z \overset{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$

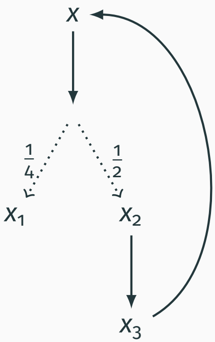
$$x +_p y \overset{(C_p)}{=} y +_{1-p} x$$

$$x +_p x \overset{(I_p)}{=} x$$

$$(\mathcal{D}(X), CS_p(\_, \_)) \cong (Terms(X, \Sigma)_{/E}, +_p)$$

EXAMPLE: PROBABILITY+TERMINATION (SUBDISTRIBUTIONS)

Monad $(\mathcal{M}, \eta, \mu)$ in Set	$\Longleftrightarrow$	Equational Theory (Σ, E) for Σ a signature, E a set of equations



$$\tau : X \to \mathcal{D}^{\leq}(X)$$

$$\tau(x) = \tfrac{1}{4}\, x_1 + \tfrac{1}{2}\, x_2$$
$$\tau(x_1) = \overline{0}$$
...

subdistribution = $\sum_i p_i\, x_i$ with $\sum_i p_i \leq 1$

EXAMPLE: PROBABILITY+TERMINATION (SUBDISTRIBUTIONS)

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Subdistribution monad
$(\mathcal{D}^{\leq}, \eta, \mu)$

- $\mathcal{D}^{\leq} : X \mapsto \{\Delta \mid \Delta$ is a finitely supported probability subdistribution on $X\}$

$\Longleftrightarrow$

Equational theory of
pointed convex algebras

- $\Sigma = \star$ and $+_p$ for all $p \in (0,1)$
- axioms of E =

$$(x +_q y) +_p z \stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$

$$x +_p y \stackrel{(C_p)}{=} y +_{1-p} x$$

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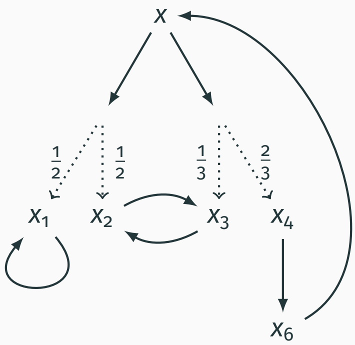
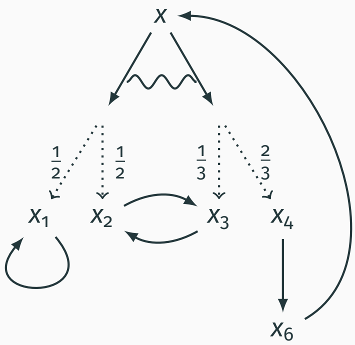


- a transition reaches a set of probability distributions $\{\ \frac{1}{2}x_1 + \frac{1}{2}x_2,\ \ \frac{1}{3}x_3 + \frac{2}{3}x_4\ \}$
- Problem: $\mathcal{P} \circ \mathcal{D}$ is not a monad [Varacca, Winskel 2006]



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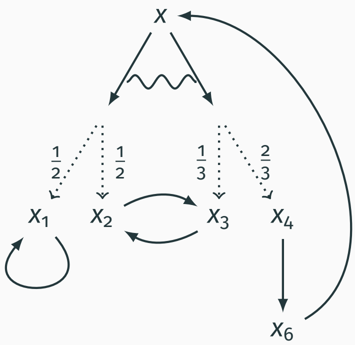
Solution: use **convex sets of probability distributions**

$$\{\,\frac{1}{2}x_1 + \frac{1}{2}x_2, \ldots, \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{6}x_3 + \frac{1}{3}x_4, \ldots, \frac{1}{3}x_3 + \frac{2}{3}x_4\,\}$$



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+ accounts for probabilistic schedulers

The monad $(\mathcal{C}, \eta, \mu)$ in Set:

- $\mathcal{C} : X \mapsto \{S \mid S$ is a non-empty, convex-closed, finitely generated set of finitely supported probability distributions over $X\}$

- $\eta_X : X \to \mathcal{C}(X)$

$$\eta_X : x \mapsto \{ 1x \}$$

- $\mu_X : \mathcal{CC}(X) \to \mathcal{C}(X)$

$$\mu_X : \bigcup_i \{\Delta_i\} \mapsto \bigcup_i \mathtt{WMS}(\Delta_i)$$

with $\mathtt{WMS} : \mathcal{DC}(X) \to \mathcal{C}(X)$ the *weighted Minkowski sum*

$$\mathtt{WMS}(\sum_{i=1}^{n} p_i S_i) = \{\sum_{i=1}^{n} p_i \cdot \Delta_i \mid \text{for each } 1 \leq i \leq n, \Delta_i \in S_i\}$$

[Jacobs 2008 ...]

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Convex sets (non-empty) of distributions monad $\mathcal{C}(X) = \{S \mid S$ is a non-empty, convex-closed, finitely generated set of finitely supported probability distributions over $X\}$

$\Longleftrightarrow$

Equational theory of convex semilattices

- Σ = $\oplus$ and $+_p$ for all $p \in (0, 1)$
- axioms E :
  - axioms of semilattices
  - axioms of convex algebras
  - distributivity axiom (D)
    $(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$

[Bonchi, Sokolova, V. 2019 and 2021]

EQUATIONAL THEORY FOR NONDETERMINISM AND PROBABILITY

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Convex sets (non-empty) of distributions monad
$\mathcal{C}(X) = \{S \mid S$ is a non-empty, convex-closed, finitely generated set of finitely supported probability distributions over $X\}$

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[Bonchi, Sokolova, V. 2019 and 2021]

$$\left(\mathcal{C}(X), \uplus^{cc}, WMS_p(\_, \_)\right) \cong \left(Terms(X, \Sigma)_{/E}, \oplus, +_p\right)$$

Monad $(\mathcal{M}, \eta, \mu)$
in Set

$\Longleftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Convex sets
(possibly empty)
of distributions
monad $\mathcal{C}^{\emptyset}$

$\Longleftrightarrow$

Equational theory of convex semilattices
with bottom and black-hole

- $\Sigma = \star, \oplus, +_p$ for all $p \in (0, 1)$
- axioms of $E =$
  - axioms of convex semilattices
  - bottom axiom $x \oplus \star = x$
  - black-hole axiom $x +_p \star = \star$

Monad $(\mathcal{M}, \eta, \mu)$ in Set $\Longleftrightarrow$ Equational Theory (Σ, E) for Σ a signature, E a set of equations

$\perp$-closed convex sets (possibly empty) of subdistributions monad $\mathcal{C}^{\downarrow}$ $\Longleftrightarrow$ Equational theory of convex semilattices with bottom

- $\Sigma = \star, \oplus, +_p$ for all $p \in (0, 1)$
- axioms E :
  - axioms of convex semilattices
  - bottom axiom $\quad x \oplus \star = x$

[Mio, Sarkis, V. 2021]

- subdistribution = $\sum_i p_i x_i$ with $\sum_i p_i \leq 1$
- S is $\perp$-closed = if $\sum_i p_i x_i \in S$ then $\sum_i q_i x_i \in S$ with $q_i \leq p_i$

APPLICATION: REASONING ON EQUIVALENCE OF TRANSITION SYSTEMS

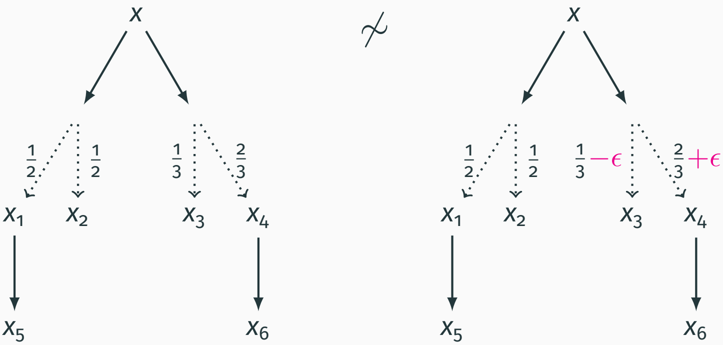
For transition systems with nondeterminism, probabilities, termination, combinations...

- axiomatizations and equational reasoning for bisimulation equivalence

$$x \sim y \quad \text{iff} \quad x = y \text{ in the equational theory}$$

- proof techniques for trace equivalence (via powerset construction)

[Bonchi, Pous 2013], [Bonchi, Sokolova, V. 2019]...


x
1/2
1/2
1/3
2/3
x1 x2 x3 x4
x5 x6
≁
x
1/2
1/2
1/3 −ϵ
2/3 +ϵ
x1 x2 x3 x4
x5 x6

Monads on Metric Spaces and Quantitative Equational Theories

Computational effect
(monad in Set)

$\Leftrightarrow$

Equational Theory (Σ, E)
for Σ a signature, E a set of equations

Effect+distance
(monad in Met)

$\Leftrightarrow$

Quantitative Equational Theory (Σ, Q)
for Σ a signature, Q quantitative inferences

Monad $(\mathcal{M}, \eta, \mu)$
in Set

- functor $\mathcal{M} : X \mapsto \mathcal{M}(X)$
- unit $\eta_X : X \to \mathcal{M}(X)$
- multiplication $\mu_X : \mathcal{M}(\mathcal{M}(X)) \to \mathcal{M}(X)$

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$
in Met

Metric Space (X, d)

- X a set
- $d : X \times X \to [0, 1]$ a metric on X

- functor $\hat{\mathcal{M}} : (X, d) \mapsto (\mathcal{M}(X), \mathrm{lift}_{\mathcal{M}}(d))$
  with $\mathrm{lift}_{\mathcal{M}}$: metric on $X \mapsto$ metric on $\mathcal{M}(X)$
- unit and multiplication are non-expansive

The powerset monad $(\mathcal{P}, \eta, \mu)$ can be lifted to a monad $(\hat{\mathcal{P}}, \hat{\eta}, \hat{\mu})$ in Met:

- $\hat{\mathcal{P}} : (X, d) \mapsto (\mathcal{P}(X), H(d))$

$H(d) =$ Hausdorff lifting of d

The powerset monad $(\mathcal{P}, \eta, \mu)$ can be lifted to a monad $(\hat{\mathcal{P}}, \hat{\eta}, \hat{\mu})$ in Met:

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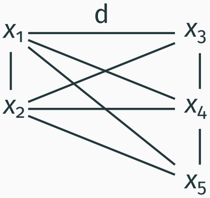

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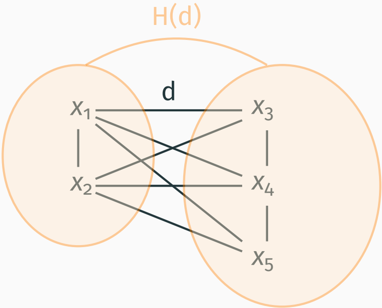

$$H(d)(S_1, S_2) = \max\left\{ \sup_{x \in S_1} \inf_{y \in S_2} d(x,y) \ , \ \sup_{y \in S_2} \inf_{x \in S_1} d(x,y) \right\}$$

The powerset monad $(\mathcal{P}, \eta, \mu)$ can be lifted to a monad $(\hat{\mathcal{P}}, \hat{\eta}, \hat{\mu})$ in Met:

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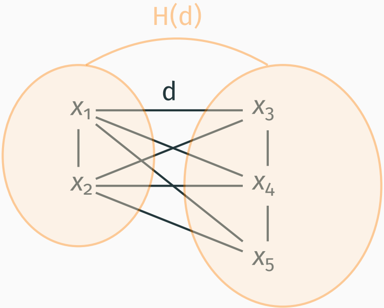

- $\hat{\eta}_{(X,d)} : (X, d) \to (\mathcal{P}(X), H(d))$ and $\hat{\mu}_{(X,d)} : \big(\mathcal{P}\mathcal{P}(X), H(H(d))\big) \to (\mathcal{P}(X), H(d))$ non-expansive

The distribution monad $(\mathcal{D}, \eta, \mu)$ can be lifted to a monad $(\hat{\mathcal{D}}, \hat{\eta}, \hat{\mu})$ in Met:

- $\hat{\mathcal{D}} : (X, d) \mapsto (\mathcal{D}(X), K(d))$

$K(d) =$ Kantorovich lifting of d

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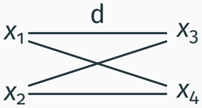

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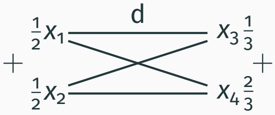

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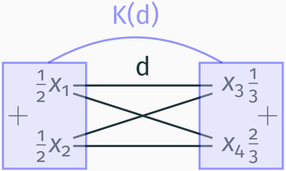


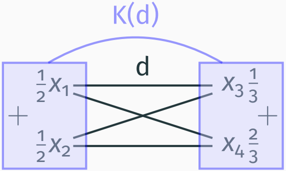
$$K(d)(\Delta_1, \Delta_2) = \inf_{\omega \in Coup(\Delta_1, \Delta_2)} \Big(\sum_{(x_1, x_2) \in X \times X} \omega(x_1, x_2) \cdot d(x_1, x_2) \Big)$$

with $Coup(\Delta_1, \Delta_2)$ the set of couplings of Δ_1 and Δ_2, i.e., probability distributions on $X \times X$ such that the marginals of ω are Δ_1 and Δ_2

The distribution monad $(\mathcal{D}, \eta, \mu)$ can be lifted to a monad $(\hat{\mathcal{D}}, \hat{\eta}, \hat{\mu})$ in Met:

- $\hat{\mathcal{D}} : (X, d) \mapsto (\mathcal{D}(X), K(d))$

$K(d) =$ Kantorovich lifting of d



- $\hat{\eta}_{(X,d)} : (X, d) \to (\mathcal{D}(X), K(d))$ and $\hat{\mu}_{(X,d)} : \big(\mathcal{D}\mathcal{D}(X), K(K(d))\big) \to (\mathcal{D}(X), K(d))$ non-expansive

The monad $(\mathcal{C}, \eta, \mu)$ of convex sets of distributions can be lifted to a monad $(\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})$ in Met:

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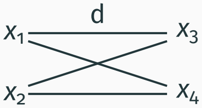

THE MONAD OF CONVEX SETS OF DISTRIBUTIONS, ON METRIC SPACES

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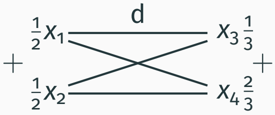

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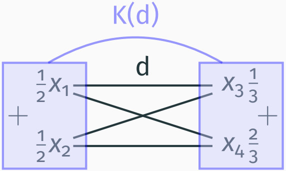

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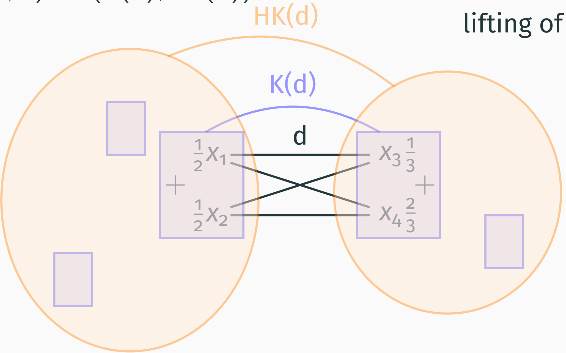

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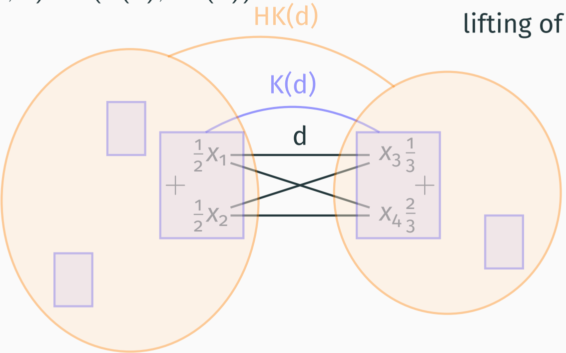

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  non-expansive

Computational effect (monad in Set)	$\Leftrightarrow$	Equational Theory (Σ, E) for Σ a signature, E a set of equations
⇣		⇣
Effect+distance (monad in Met)	$\Leftrightarrow$	Quantitative Equational Theory (Σ, Q) for Σ a signature, Q quantitative inferences

Signature Σ = set of operations *op*, each with its arity

- terms $t := x | op(t_1, ...t_n) \qquad \forall op \in \Sigma$
- quantitative equations $t =_\varepsilon s$
- Q a set of quantitative inferences $\{t_i =_{\varepsilon_i} s_i\}_{i \in I} \vdash t =_\varepsilon s$

Deductive system of quantitative equational logic

(Reflexivity) $\emptyset \vdash t =_0 t$

(Symmetry) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triangular) $\{t =_{\varepsilon_1} u, u =_{\varepsilon_2} s\} \vdash t =_{\varepsilon_1 + \varepsilon_2} s$

[Mardare, Panangaden, Plotkin 2016...]

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Models: quantitative algebras (A, Σ^A, d_A) satisfying Q

$$t =_\varepsilon s \quad \text{is satisfied if} \quad \forall \iota : X \to A,\ d_A([\![t]\!]_A^\iota, [\![s]\!]_A^\iota) \leq \varepsilon$$

Quantitative algebra of terms modulo equations:

$$(Terms(X, \Sigma)_{/Q}, \Sigma, d_{(\Sigma,Q)})$$

$$\text{with } d_{(\Sigma,Q)} = (t, t') \mapsto \inf\{\varepsilon \mid \emptyset \vdash t =_\varepsilon t'\}$$

[Mardare, Panangaden, Plotkin 2016...]

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$
in Met

$\Longleftrightarrow$

Quantitative Equational Theory (Σ, Q)
for Σ a signature, Q quantitative inferences

(Σ, Q) **is a presentation of** $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$

The category **EM**$(\hat{\mathcal{M}})$ of Eilenberg-Moore algebras for $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ is isomorphic to the category **QA**(Σ, Q) of quantitative (Σ, Q)-algebras

Corollary: equational reasoning on free objects

Free quantitative algebra for the monad $\cong (Terms(X, \Sigma)_{/Q}, \Sigma, d_{(\Sigma,Q)})$

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met $\Leftrightarrow$ Quantitative Equational Theory (Σ, Q) for Σ a signature, Q quantitative inferences

Powerset (non-empty) monad in Met, with Hausdorff lifting $\Leftrightarrow$

Quantitative equational theory of semilattices

- $\Sigma = \oplus$
- quantitative inferences $Q =$

- axioms of semilattices, with $t = t'$ becoming $\emptyset \vdash t =_0 t'$
- $\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\} \vdash x_1 \oplus x_2 =_{\max(\epsilon_1,\epsilon_2)} y_1 \oplus y_2$

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$$\left(\mathcal{P}(X), \bigcup, H(d)\right) \cong \left(Terms(X,\Sigma)_{/Q}, \oplus,\ d_{(\Sigma,Q)}\right)$$

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met $\Leftrightarrow$ Quantitative Equational Theory (Σ, Q) for Σ a signature, Q quantitative inferences

Distribution monad in Met, with Kantorovich lifting $\Leftrightarrow$ Quantitative equational theory of convex algebras

- $\Sigma = +_p$ for all $p \in (0, 1)$
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Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met $\Leftrightarrow$ Quantitative Equational Theory (Σ, Q) for Σ a signature, Q quantitative inferences

Convex sets (non-empty) of distributions monad in Met, with Hausdorff–Kantorovich lifting $\Leftrightarrow$ Quantitative equational theory of convex semilattices

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[Mio, V. 2020]

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[Mio, V. 2020]

$$\left(\mathcal{C}(X), \uplus, WMS_p(\_, \_), HK(d)\right) \cong \left(Terms(X, \Sigma)_{/Q}, \oplus, +_p, d_{(\Sigma, Q)}\right)$$

RECAP: ADDING TERMINATION, IN SETS

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met $\Longleftrightarrow$ Equational Theory (Σ, E) for Σ a signature, E a set of equations

Convex sets (possibly empty) of distributions monad $\mathcal{C}^{\emptyset}$ $\Longleftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$ and black-hole $x +_p \star = \star$

$\perp$-closed convex sets (possibly empty) of subdistributions monad $\mathcal{C}^{\perp}$ $\Longleftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$

Convex sets (possibly empty) of distributions monad $\mathcal{C}^{\emptyset}$	$\Leftrightarrow$	Equational theory of convex semilattices with bottom $x \oplus \star = x$ and black-hole $x +_p \star = \star$

Negative results in Met:

- The quantitative equational theory of convex semilattices with bottom and black-hole is trivial
- The multiplication μ of $\mathcal{C}^{\emptyset}$ is not non-expansive $\Rightarrow$ the same monad cannot be lifted to Met

[Mio, Sarkis, V. 2021]

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$\perp$-closed convex sets (possibly empty) of subdistributions monad $\mathcal{C}^{\downarrow}$	$\Leftrightarrow$	Equational theory of convex semilattices with bottom $x \oplus \star = x$

[Mio, Sarkis, V. 2021]

Convex sets (possibly empty) of distributions monad $\mathcal{C}^{\emptyset}$ $\Leftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$ and black-hole $x +_p \star = \star$

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$\perp$-closed convex sets (possibly empty) of subdistributions monad in Met $\mathcal{C}^{\downarrow}$ with HK $\Leftrightarrow$ Quantitative equational theory of convex semilattices with bottom $x \oplus \star = x$

[Mio, Sarkis, V. 2021]

RECAP

YES in Met	Convex sets (non-empty) of distributions monad $\mathcal{C}$	$\Leftrightarrow$	Equational theory of convex semilattices
NO in Met	Convex sets (possibly empty) of distributions monad $\mathcal{C}^{\emptyset}$	$\Leftrightarrow$	Equational theory of convex semilattices with bottom $x \oplus \star = x$ and black-hole $x +_p \star = \star$
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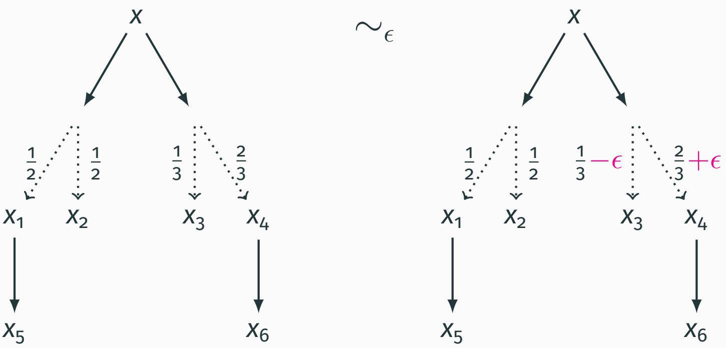

A sound and complete proof technique for bisimulation distance

$x \sim_\epsilon y$ iff $x =_\epsilon y$ in the quantitative equational theory

[Mio, Sarkis, V. 2021]

Varying the liftings

Different ways of lifting a metric d to probability distributions $\mathcal{D}(X)$

- Kantorovich lifting on probability distributions

$$K(d)(\Delta_1, \Delta_2) = \inf_{\omega \in Coup(\Delta_1,\Delta_2)} \Big(\sum_{(x_1,x_2)\in X\times X} \omega(x_1, x_2) \cdot d(x_1, x_2) \Big)$$

  with $Coup(\Delta_1, \Delta_2)$ the set of couplings of Δ_1 and Δ_2, i.e., probability distributions on $X \times X$ such that the marginals of ω are Δ_1 and Δ_2

- Łukaszyk–Karmowski lifting on probability distributions

$$ŁK(d)(\Delta_1, \Delta_2) = \sum_{x\in supp(\Delta_1)} \sum_{y\in supp(\Delta_2)} \Delta_1(x) \cdot \Delta_2(y) \cdot d(x, y)$$

[Castro et al. 2021]

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A metric? Presented by a quantitative equational theory ?

$(X, d : X \times X \to [0, 1])$ is a metric space iff

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$
4. $d(x, y) = 0 \ \Rightarrow \ x = y$

For (X, d) a metric space, $(\mathcal{D}(X), ŁK(d))$ is not a metric space

$\exists \Delta$ such that $ŁK(d)(\Delta, \Delta) > 0$

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For (X, d) a metric space, $(\mathcal{D}(X), ŁK(d))$ is not a metric space

$\exists \Delta$ such that $ŁK(d)(\Delta, \Delta) > 0$

Solution: generalised metric spaces

(X, d) with d a function $d : X \times X \to [0, 1]$ (aka “fuzzy relation”)

d may satisfy a subset of:

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$
4. $d(x, y) = 0 \Rightarrow x = y$
5. $d(x, z) \leq \max\{d(x, y), d(y, z)\}$

Examples:

- Metric spaces := 1 + 2 + 3 + 4
- Ultrametric spaces := 1 + 2 + 3 + 4 + 5
- Pseudo-metric spaces := 1 + 2 + 3
- Diffuse metric spaces := 2 + 3

In the deductive system of quantitative equational theories: operations are required to be nonexpansive wrt the product metric

$$s_1 =_{\varepsilon_1} t_1, \ldots, s_n =_{\varepsilon_n} t_n \vdash \mathrm{op}(s_1, \ldots, s_n) =_{max\{\varepsilon_1, \ldots, \varepsilon_n\}} \mathrm{op}(t_1, \ldots, t_n)$$

i.e., in all quantitative algebras (A, Σ^A, d_A), operations define a nonexpansive map $op^A : (A^n, \mathbf{L}_\times(d)) \to (A, d)$, where

$$\mathbf{L}_\times(d)((a_1, \ldots, a_n), (a'_1, \ldots, a'_n)) = \max_i \{d(a_i, a'_i)\}$$

In $(\mathcal{D}(X), \text{Ł}K(d))$, the operation $+_p$ is not nonexpansive wrt to the product metric, i.e., $\exists \Delta_1, \Delta_2, \Delta'_1, \Delta'_2$ such that

$$\text{Ł}K(d)(\Delta_1 +_{\frac{1}{2}} \Delta_2, \Delta'_1 +_{\frac{1}{2}} \Delta'_2) > \mathbf{L}_\times(\text{Ł}K(d))((\Delta_1, \Delta'_1), (\Delta_2, \Delta'_2))$$

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Solution: remove the nonexpansiveness requirement

Extend the framework of quantitative equational theories to include:

- generalised metric spaces
- operations which are not nonexpansive

How?

- separate equality from quantitative equality: equations and quantitative equations coexist, with relationship determined by axioms

$$x = y \quad \text{different from} \quad x =_0 y$$

- remove rule of nonexpansiveness, and allow for arbitrary operations

$\Rightarrow$ a new framework for quantitative equational reasoning, with a sound and complete deductive apparatus [Mio, Sarkis, V. 2022] effect+distance (monad in GMet) $\Leftrightarrow$ (generalised) quantitative equational theory

monads in Met seen so far $\Leftrightarrow$ (generalised) quantitative equational theories corresponding to those seen so far

distribution monad $\hat{\mathcal{D}}$ in DMet, with Łukaszyk–Karmowski lifting $\Leftrightarrow$ (generalised) quantitative equational theory

- $\Sigma = +_p$ for all $p \in (0, 1)$
- equations and quantitative inferences:
  - axioms of convex algebras,
  - quantitative axiom

$$\left\{ \begin{matrix} x_1 =_{\varepsilon_{11}} x_1, x_2 =_{\varepsilon_{21}} x_1 \\ x_1 =_{\varepsilon_{12}} y_2, y_2 =_{\varepsilon_{22}} y_2 \end{matrix} \right\} \vdash x_1 +_p x_2 =_\delta y_1 +_p y_2$$

with $\delta = p^2\varepsilon_{11} + (1-p)p\varepsilon_{21} + p(1-p)\varepsilon_{12} + (1-p)^2\varepsilon_{22}$

BISIMULATION EQUIVALENCE (PROBABILISTIC)

- Effect: probabilities $\tau : X \to \mathcal{D}(X)$
- Equivalence relation R on X such that $x\ R\ y$ implies $\tau(x)\ \hat{R}\ \tau(y)$

with $\hat{R}$ a lifting of R to $\mathcal{D}(X)$ defined as:

$$\Delta_1\ \hat{R}\ \Delta_2 \text{ iff } \forall A \in X_{/R}, \Delta_1(A) = \Delta_2(A)$$

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$$\Delta_1 \; \hat{R} \; \Delta_2 \text{ iff } \emptyset \vdash_R t_{\Delta_1} = t_{\Delta_2}$$

where $\vdash_R$ is derivability in the theory of convex algebras+equations induced by R

BISIMULATION EQUIVALENCE (NONDETERMINISM+TERMINATION)

- Effect: nondeterminism+termination $\tau : X \to \mathcal{P}^{\emptyset}(X)$
- Equivalence relation R on X such that $x \; R \; y$ implies $\tau(x) \; \hat{R} \; \tau(y)$ with $\hat{R}$ a lifting of R to $\mathcal{P}^{\emptyset}(X)$ defined as:

$$S_1 \; \hat{R} \; S_2 \text{ iff } \forall x' \in S_1 \exists y' \in S_2 \text{ s.t. } x' \; R \; y' \text{ and } \forall y' \in S_2 \exists x' \in S_1 \text{ s.t. } x' \; R \; y'$$

$$S_1 \; \hat{R} \; S_2 \text{ iff } \emptyset \vdash_R t_{S_1} = t_{S_2}$$

where $\vdash_R$ is derivability in the theory of semilattices with bottom+equations induced by R

BISIMULATION EQUIVALENCE (NONDETERMINISM+TERMINATION)

- Effect: nondeterminism+termination $\tau : X \to \mathcal{P}^{\emptyset}(X)$
- Equivalence relation R on X such that $x\ R\ y$ implies $\tau(x)\ \hat{R}\ \tau(y)$

with $\hat{R}$ a lifting of R to $\mathcal{P}^{\emptyset}(X)$ defined as:

$S_1\ \hat{R}\ S_2$ iff $\forall x' \in S_1 \exists y' \in S_2$ s.t. $x'\ R\ y'$ and $\forall y' \in S_2 \exists x' \in S_1$ s.t. $x'\ R\ y'$

$S_1\ \hat{R}\ S_2$ iff $\emptyset \vdash_R t_{S_1} = t_{S_2}$

where $\vdash_R$ is derivability in the theory of semilattices with bottom+equations induced by R

More generally:

- bisimulation: we lift a relation R on X to a relation $\hat{R}$ on the chosen effect over X
- by the correspondence effect/equational theory, we can reason equationally on $\hat{R}$

- Effect: probabilities $\tau : X \to \mathcal{D}(X)$
- Equivalence relation R on X such that $x \, R \, y$ implies $\tau(x) \, \hat{R} \, \tau(y)$

with $\hat{R}$ a lifting of R to $\mathcal{D}(X)$ defined as:

$\Delta_1 \, \hat{R} \, \Delta_2$ iff $\forall A \in X_{/R}, \Delta_1(A) = \Delta_2(A)$

- Effect: probabilities $\tau : X \to \mathcal{D}(X)$
- Metric d on X such that $d(x, y) \leq \varepsilon$ implies $\hat{d}(\tau(x), \tau(y)) \leq \varepsilon$

with $\hat{d}$ is a lifting of d to $\mathcal{D}(X)$ defined as: the Kantorovich lifting $K(d)$

$$K(d)(\Delta_1, \Delta_2) = \inf_{\omega \in Coup(\Delta_1, \Delta_2)} \Big(\sum_{(x_1, x_2) \in X \times X} \omega(x_1, x_2) \cdot d(x_1, x_2) \Big)$$

with $Coup(\Delta_1, \Delta_2)$ the set of couplings of Δ_1 and Δ_2, i.e., probability distributions on $X \times X$ such that the marginals of ω are Δ_1 and Δ_2

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How to reason equationally on distances?

- Effect: probabilities $\tau : X \to \mathcal{D}(X)$
- Metric d on X such that $d(x, y) \leq \varepsilon$ implies $\hat{d}(\tau(x), \tau(y)) \leq \varepsilon$

with $\hat{d}$ is a lifting of d to $\mathcal{D}(X)$ defined as: the Kantorovich lifting $K(d)$

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$$\big(\mathcal{D}(X), CS_p(\_, \_), K(d)\big) \cong \big(Terms(X, \Sigma)_{/Q}, +_p,\ d_{(\Sigma,Q)}\big)$$

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- Metric d on X such that $d(x, y) \leq \varepsilon$ implies $\hat{d}(\tau(x), \tau(y)) \leq \varepsilon$

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$$\big(\mathcal{D}(X), CS_p(\_, \_), K(d)\big) \cong \big(Terms(X, \Sigma)_{/Q}, +_p, d_{(\Sigma,Q)}\big)$$

$$K(d)(\Delta_1, \Delta_2) \leq \varepsilon \text{ iff } \emptyset \vdash_d t_{\Delta_1} =_\varepsilon t_{\Delta_2}$$

where $\vdash_d$ is derivability in the quantitative equational theory of convex algebras+quantitative equations induced by d