Comer Schemes, Relation Algebras, and the Flexible Atom Conjecture

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Relation Algebras

Relation Algebras

An *integral relation algebra* $\langle A, \wedge, \vee, \neg, 0, 1, \circ, \breve{}, 1' \rangle$ is a Boolean lattice augmented with:

- Relational composition: $\circ$ and identity atom $1'$.
- For every $a, b \in A$, $a \circ b = 0 \implies a = 0$ or $b = 0$.
- Relational converse: $\breve{}$.
- The relation algebra satisfies additional equational axioms.

When the relation algebra is understood, we write A.

Relation Algebras

Representation

- Given an abstract algebraic structure A, can we *instantiate* or *represent* A over a (finite) set, with corresponding natural operations on the set?
- Ex: Every finite group is a permutation group.
- Q: Does every finite relation algebra admit a finite representation?
- No: Point algebra.

Flexible Atom Conjecture

Theorem (Comer, 1983)

Every finite integral relation algebra with a flexible atom admits a representation over a countable set.

Conjecture (Flexible Atom Conjecture)

Every finite integral relation algebra with a flexible atom admits a representation over a finite set.

Comer's Method

Ramsey Schemes

Let U be a set, and let $m \in \mathbb{Z}^+$. An m-color *Ramsey scheme* is a partition of $U \times U$ into m sets $\mathrm{Id}, R_1, \ldots, R_{m-1}$ satisfying the following:

- $R_i^{-1} = R_i$.
- $R_i \circ R_i = R_i^c$.
- For all pairs of distinct $i, j \in [m-1]$, $R_i \circ R_j = \mathrm{Id}^c$.

The relations $\mathrm{Id}, R_1, \ldots, R_{m-1}$ naturally generate a relation algebra.

Comer's Method

Comer's Method

Comer's method (1983) provides an approach to instantiate Ramsey schemes.

- Fix $m \in \mathbb{Z}^+$.
- Want prime $p \equiv 1 \pmod{2m}$ where:
  - $U = \mathbb{F}_p$.
  - Let $X_0 \leq \mathbb{F}_p^\times$ be of order $(p-1)/m$.
  - Fix a primitive element g and write $X_i = g^i X_0 = \{g^{am+i} : a \in \mathbb{Z}^+\}$.
  - The cosets satisfy some *nice* conditions.
  - Now $R_i = \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : x - y \in X_i\}$.
- Only need to check $p < m^4 + 5$.

Comer's Method

Comer Schemes

- Can we relax the notion of a Ramsey scheme and still obtain representations via Comer's method?
- Fix $m \in \mathbb{Z}^+$ and a set U. An m-color *Comer scheme* is a partition of $U \times U$ into $m + 1$ binary relations $\mathrm{Id}, R_0, \ldots, R_{m-1}$.
- We will be particularly interested in Comer schemes that also satisfy the following: for all distinct $i, j, k \in [m-1]$, $R_i \circ R_j$ contains R_k.

Comer's Method

Question

Which integral relation algebras admit representations via Comer schemes?

Definition

Let $\mathcal{A}_n([i, i+j, i+\ell])$ denote the integral relation algebra with n symmetric diversity atoms $a_0, \ldots, a_{n-1}$ and forbidden cycles of the form $\{a_i a_{i+j} a_{i+\ell} : 0 \leq i < n\}$, with the indices considered modulo n.

Comer's Method

Definition

The relation algebra 34_{65} has four symmetric atoms $1'$, a, b, c, with a flexible and forbidden cycles bbc, ccb.

Comer's Method

Theorem (Alm–Andrews–L., 2023)

The relation algebra 34_{65} *admits a finite representation over a set of size* 3697*.*

Proof.

- Abstractly, 34_{65} embeds into $\mathcal{A}_n([i, i, i + n/2])$ for all even $n \geq 6$.
- We obtain (via computer search) an instantiation via Comer's method using $p = 3697$ and $n = 24$.

$\square$

Open Questions

Conjecture (Strong Flexible Atom Conjecture)

Every finite integral relation algebra with a flexible atom is representable over a Comer scheme.

Question

The relation algebra 33_{65} *has atoms* $1', a, b, c$*, with a flexible and forbidden cycles* ccc, bcc, cbb*. Does* 33_{65} *admit a finite representation? This is the last RA in the family* N_{65} *with a flexible atom that is not known to admit a finite representation.*