

Duoidally Enriched Freyd Categories

RAMiCS'23, Augsburg, Germany

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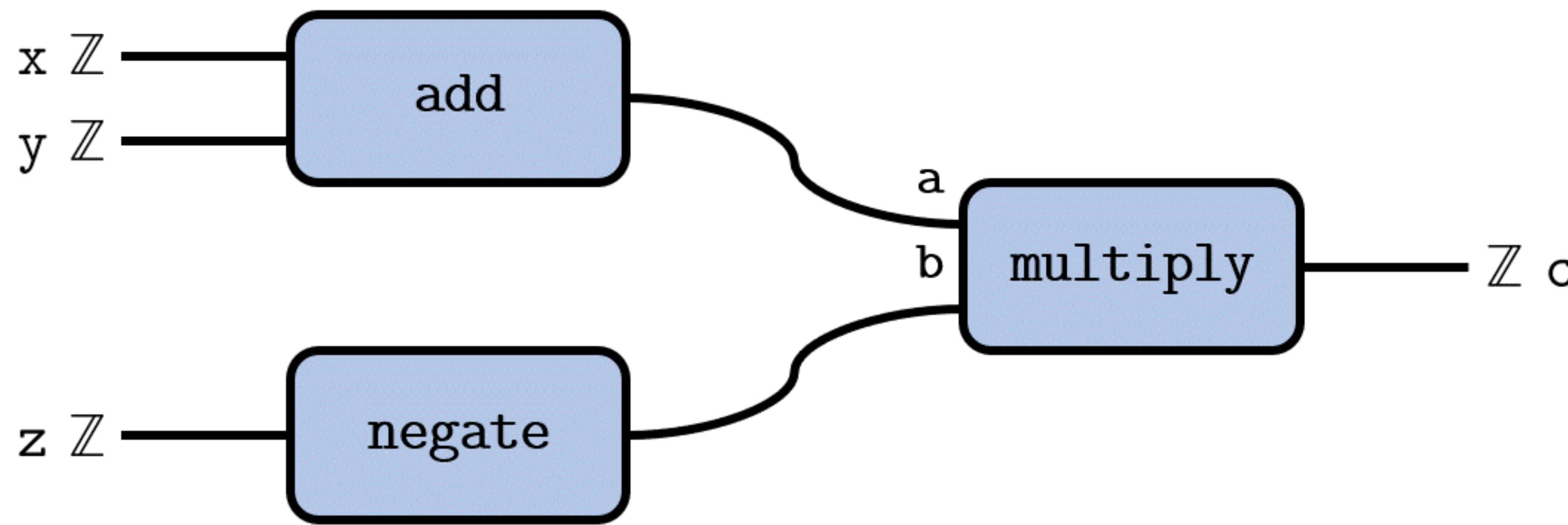
University of Edinburgh, U.K.

Roadmap

- Motivation
- Background definitions
- Duoidally enriched Freyd categories
- Examples
- More from the paper

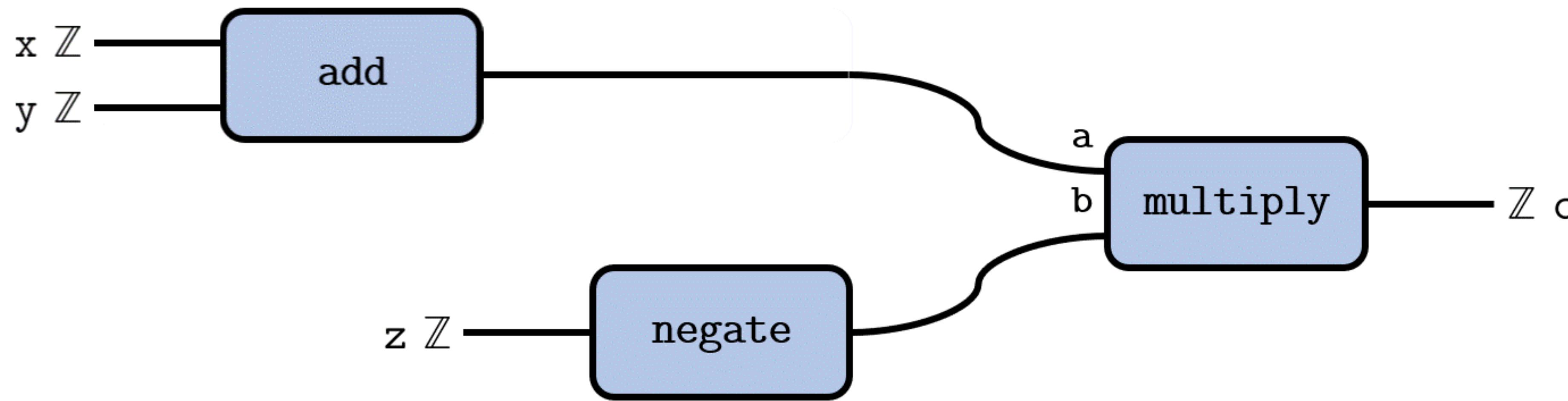
Motivation

Motivation



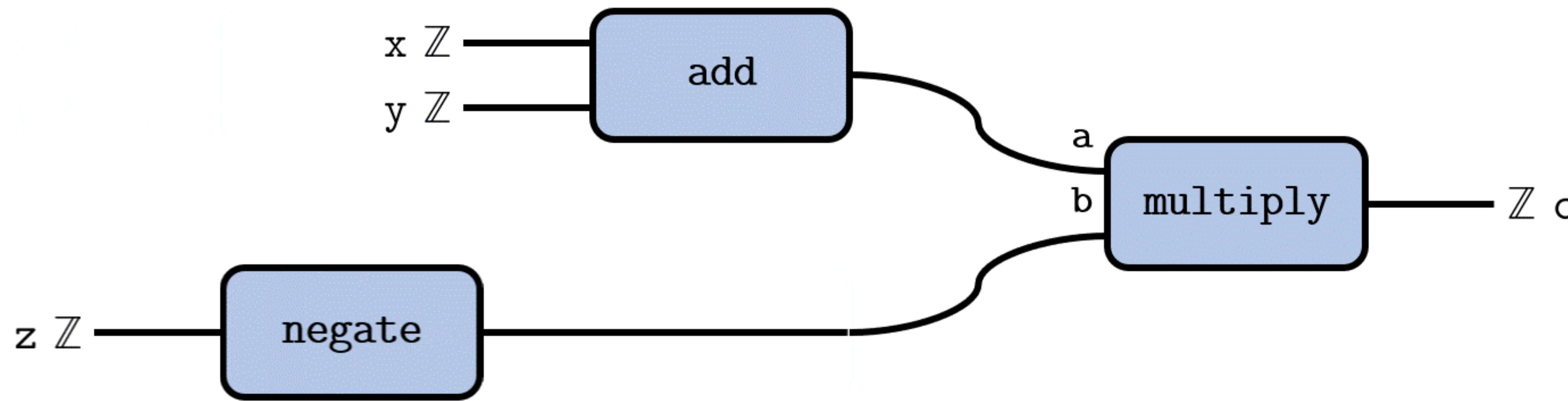
```
let (a, b) = (add(x, y), negate(z)) in  
let      c = multiply(a, b) in  
c
```

Motivation



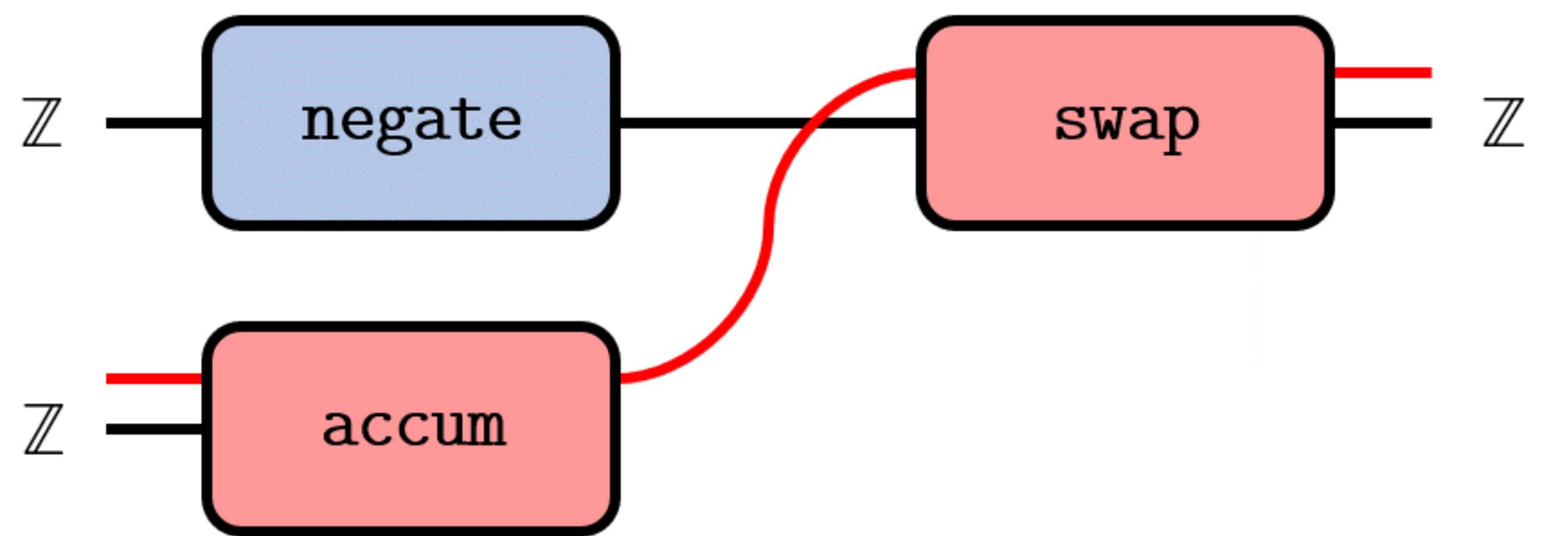
```
let a = add(x, y) in  
let b = negate(z) in  
let c = multiply(a, b) in  
c
```

Motivation



```
let b = negate(z) in  
let a = add(x, y) in  
let c = multiply(a, b) in  
c
```

Motivation



effect-free

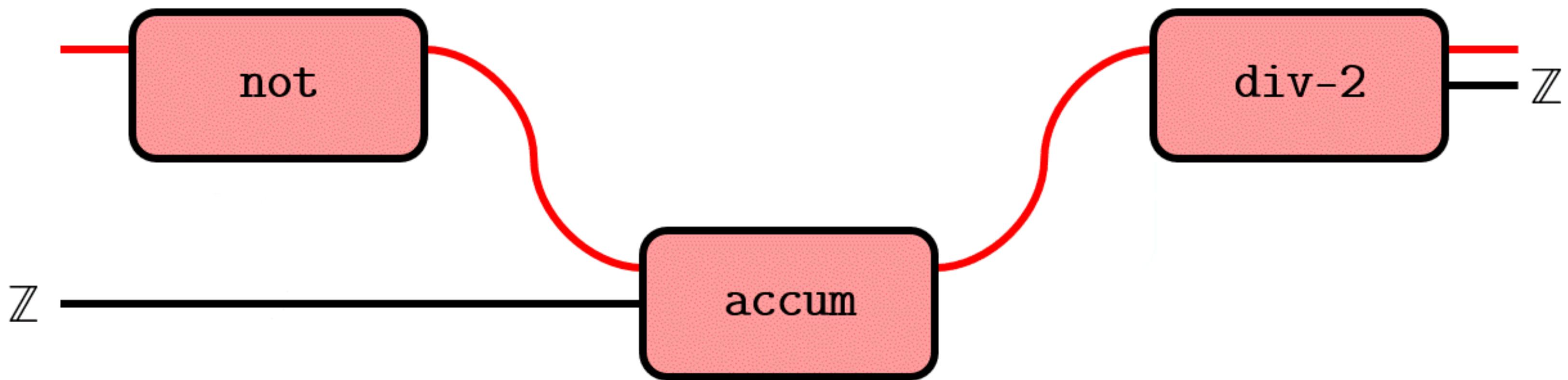
effectful

```
negate : Z -> Z  
negate(x) = -x
```

```
accum : Z -> ()  
accum(y) =  
let s = get() in  
put(y + s)
```

```
swap : Z -> Z  
swap(z) =  
let s = get() in  
put(z);  
s
```

Motivation



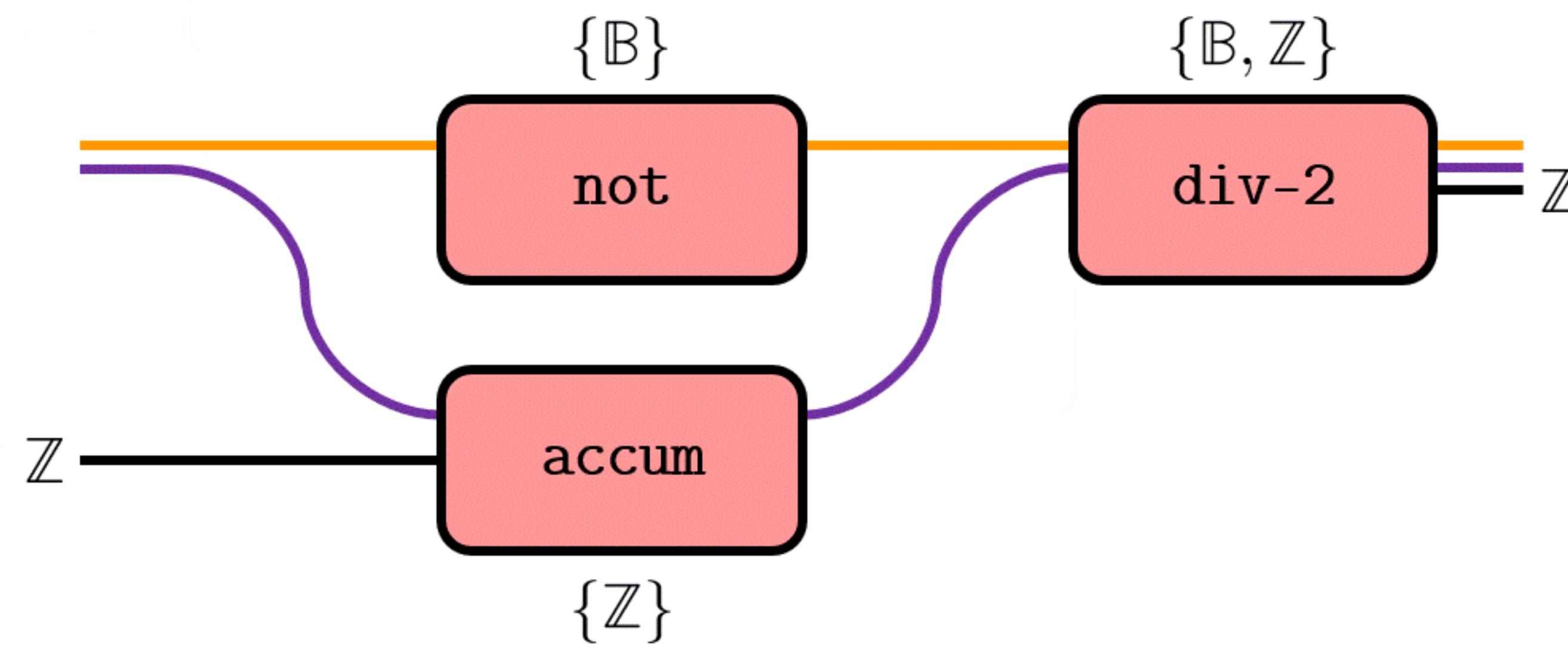
```
not : () -> ()
not() =
let (b, s) = get() in
put(¬b, s)
```

```
accum : ℤ -> ()
accum(y) =
let (b, s) = get() in
put(b, y + s)
```

```
div-2 : () -> ℤ
div-2() =
let (b, s) = get() in
if b then
  ceil(s/2)
else
  floor(s/2)
```

effectful not parallel

Motivation



```
not : () -> ()
not() =
  let (b, s) = get() in
  put(¬b, s)
```

```
accum : Z -> ()
accum(y) =
  let (b, s) = get() in
  put(b, y + s)
```

```
div-2 : () -> Z
div-2() =
  let (b, s) = get() in
  if b then
    ceil(s/2)
  else
    floor(s/2)
```

effectful in parallel

Binoidal Categories

John C. Baez
University of California, Riverside

Joint work with Mike Stay
and John Huerta

Based on joint work with
Mike Stay and John Huerta

Based on joint work with
Mike Stay and John Huerta

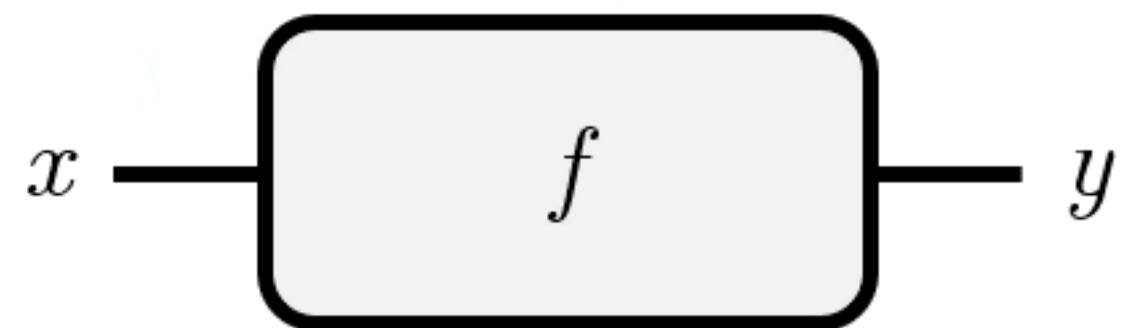
Based on joint work with
Mike Stay and John Huerta

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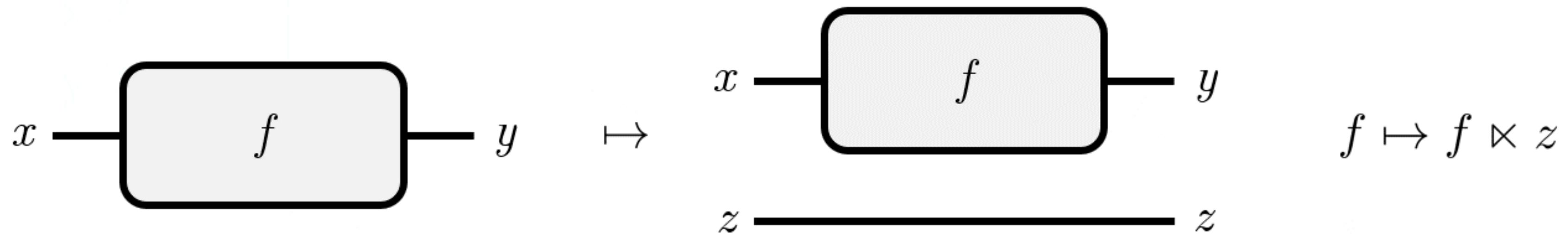
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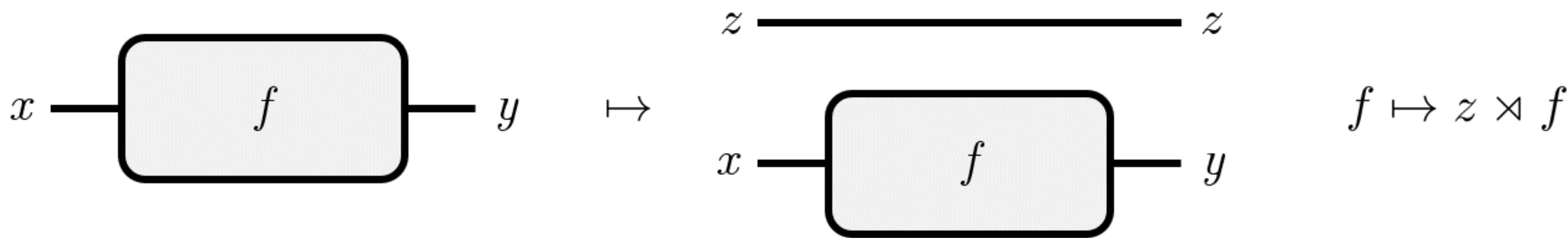
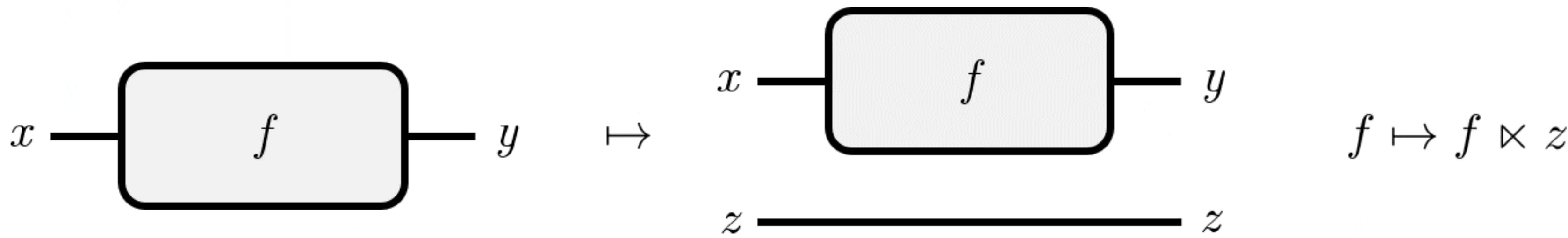
Binoidal Categories



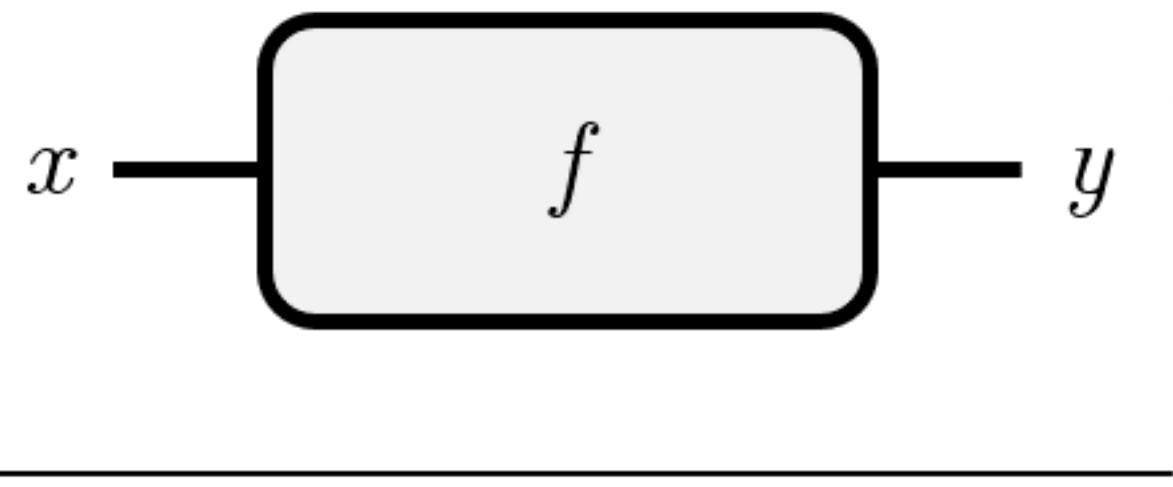
Binoidal Categories



Binoidal Categories



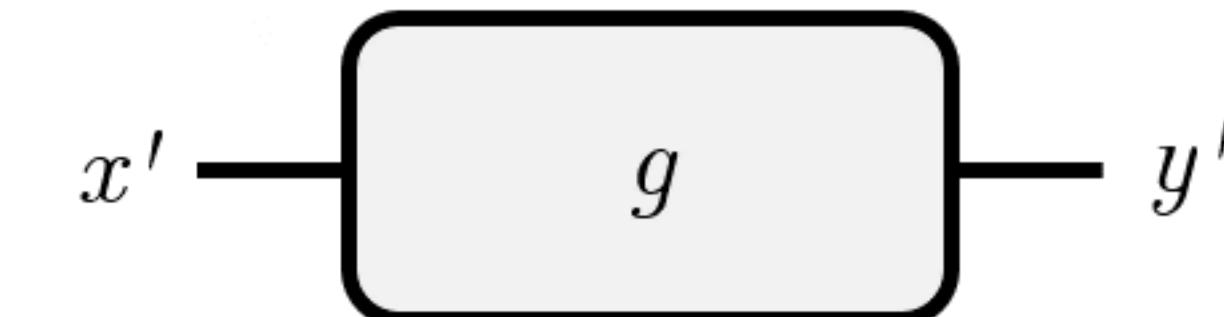
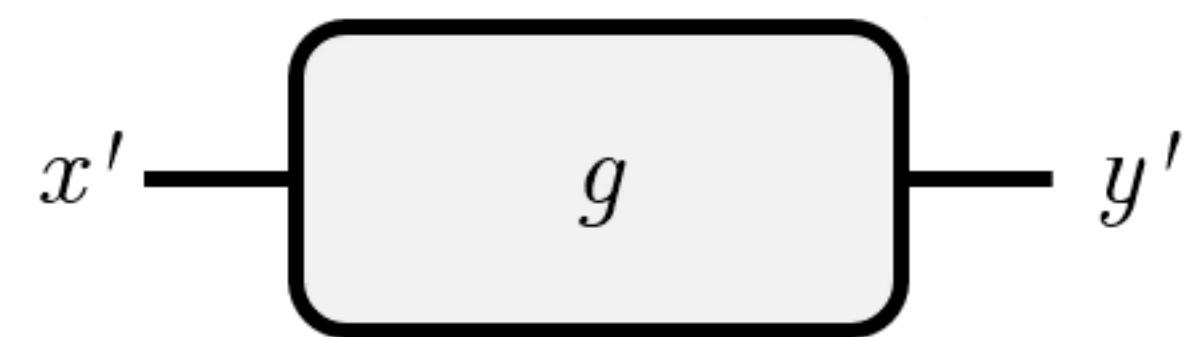
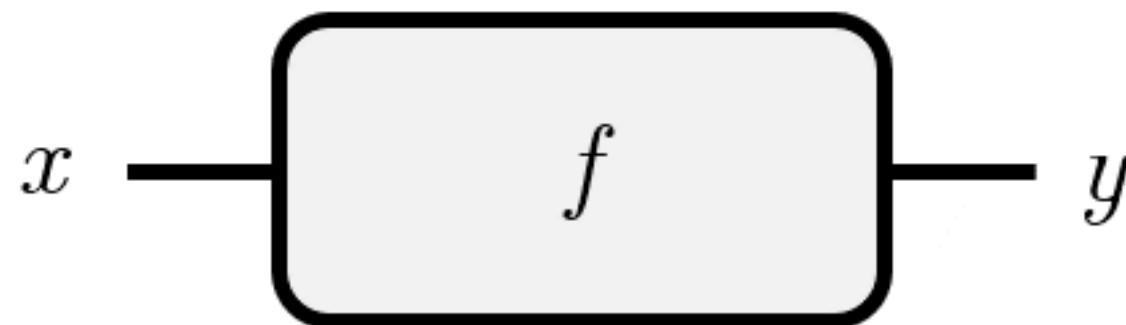
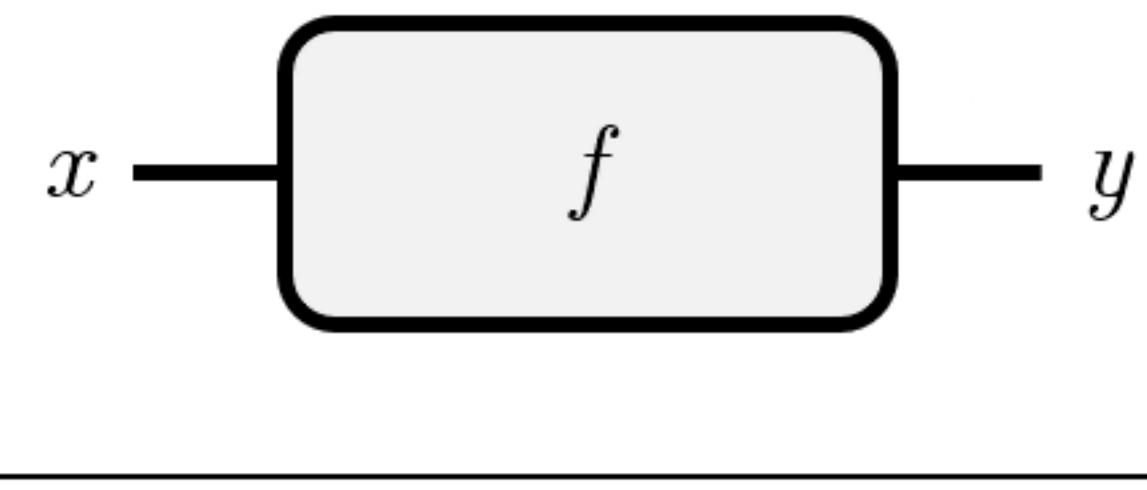
Binoidal Categories



$$x \xrightarrow{f} y \quad \mapsto \quad \begin{array}{c} x \xrightarrow{f} y \\ z \xrightarrow{\quad\quad\quad} z \end{array} \quad f \mapsto f \ltimes z$$

$$x \xrightarrow{f} y \quad \mapsto \quad \begin{array}{c} z \xrightarrow{\quad\quad\quad} z \\ x \xrightarrow{f} y \end{array} \quad f \mapsto z \rtimes f$$

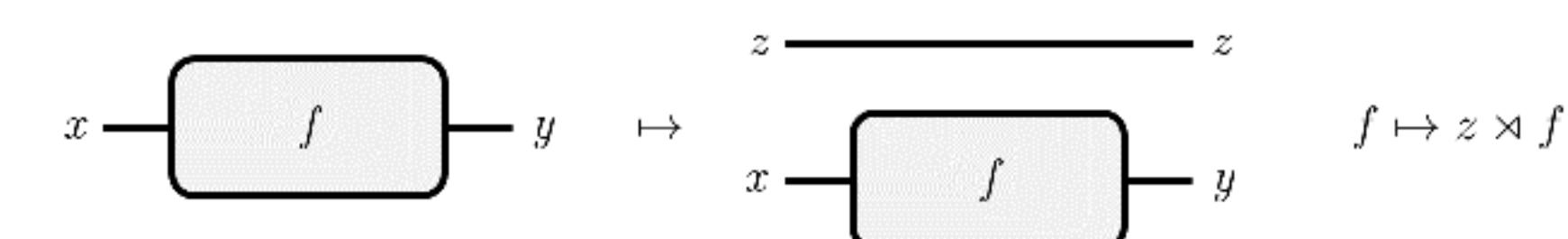
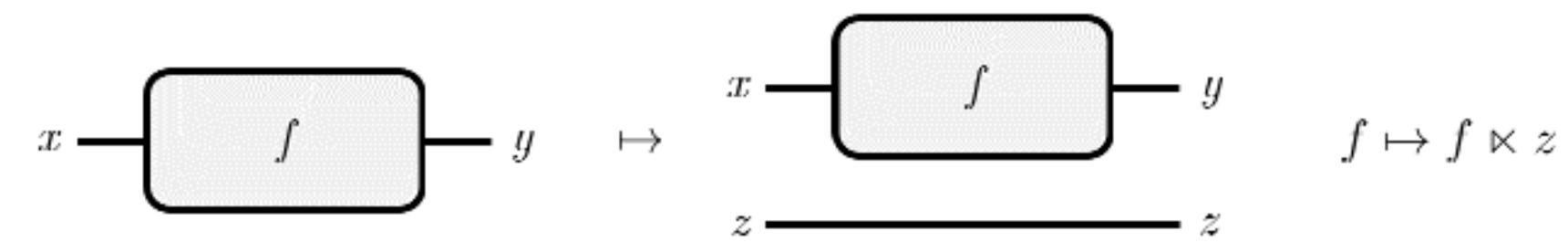
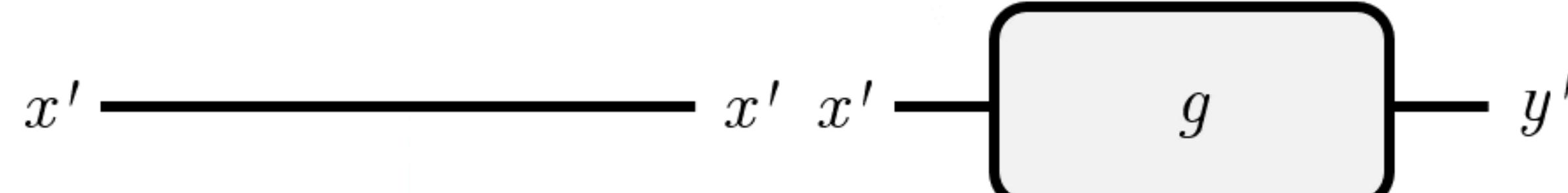
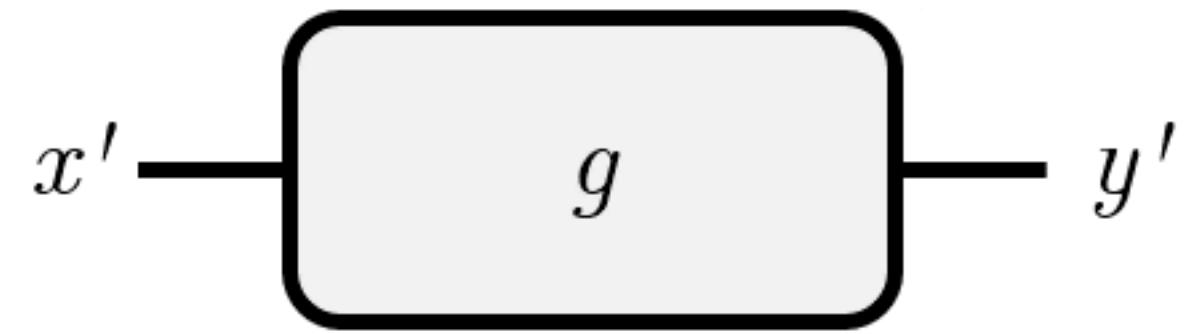
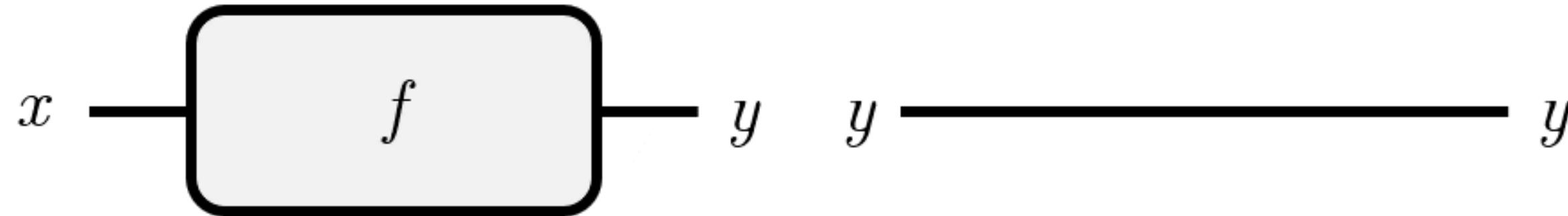
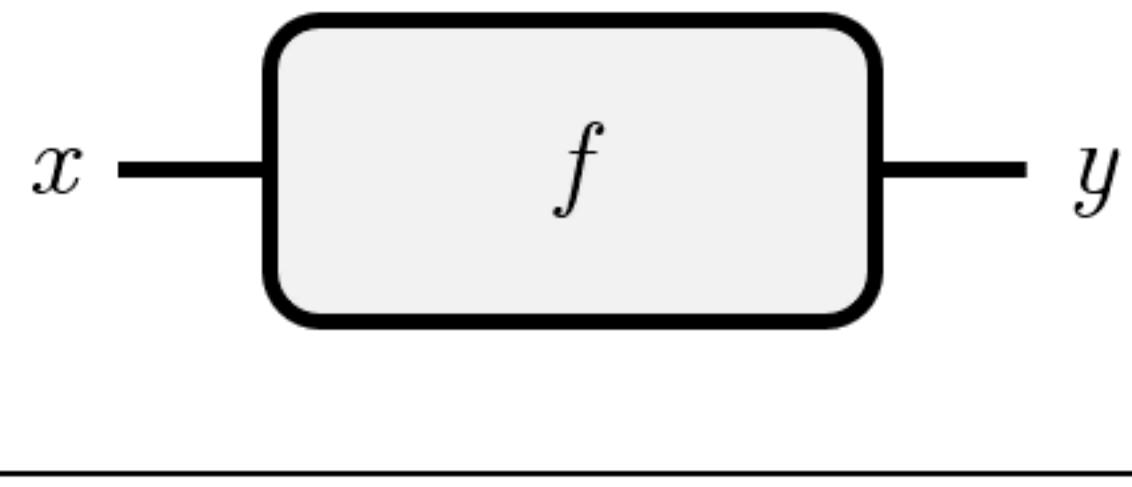
Binoidal Categories



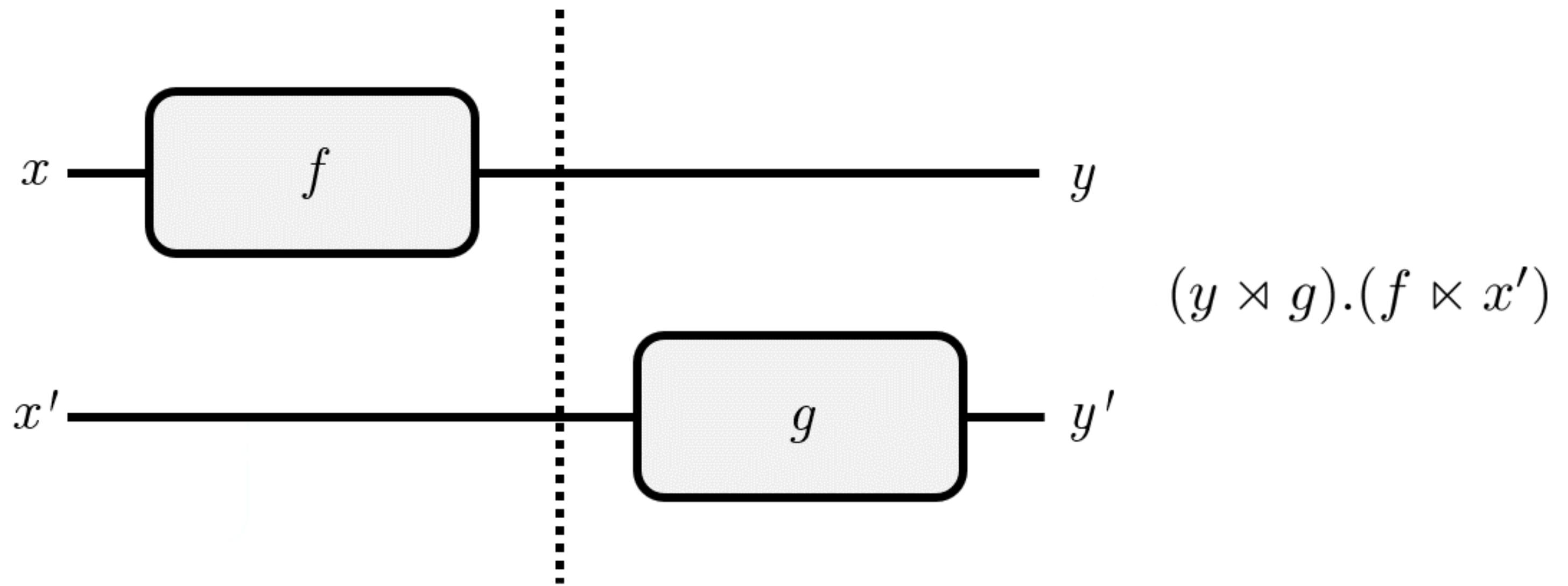
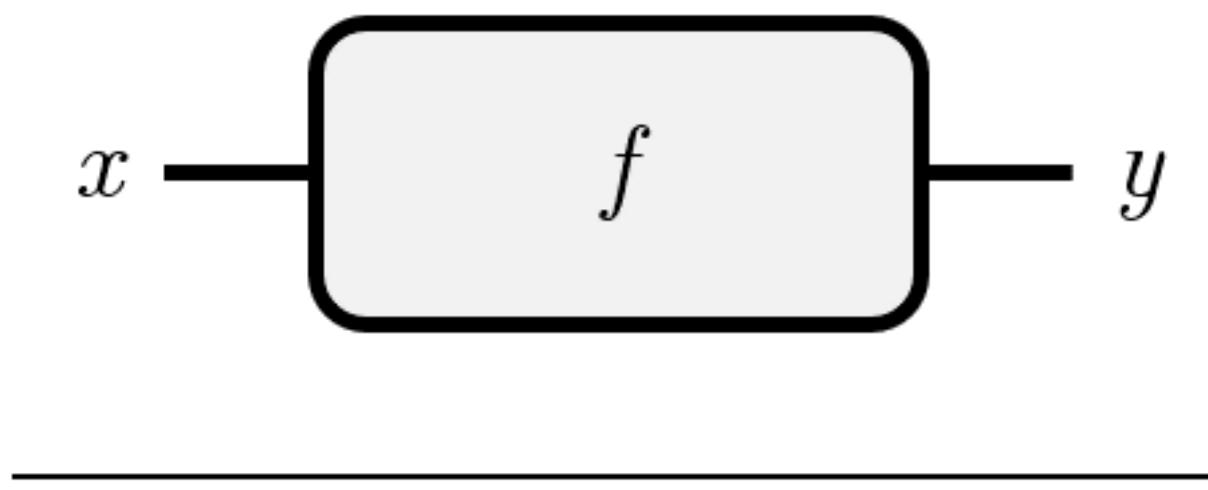
$$x \xrightarrow{f} y \mapsto \begin{array}{c} x \xrightarrow{f} y \\ z \xrightarrow{\quad} z \end{array} \quad f \mapsto f \ltimes z$$

$$\begin{array}{c} x \xrightarrow{f} y \\ z \xrightarrow{\quad} z \end{array} \mapsto x \xrightarrow{f} y \quad z \mapsto z \rtimes f \quad f \mapsto z \rtimes f$$

Binoidal Categories



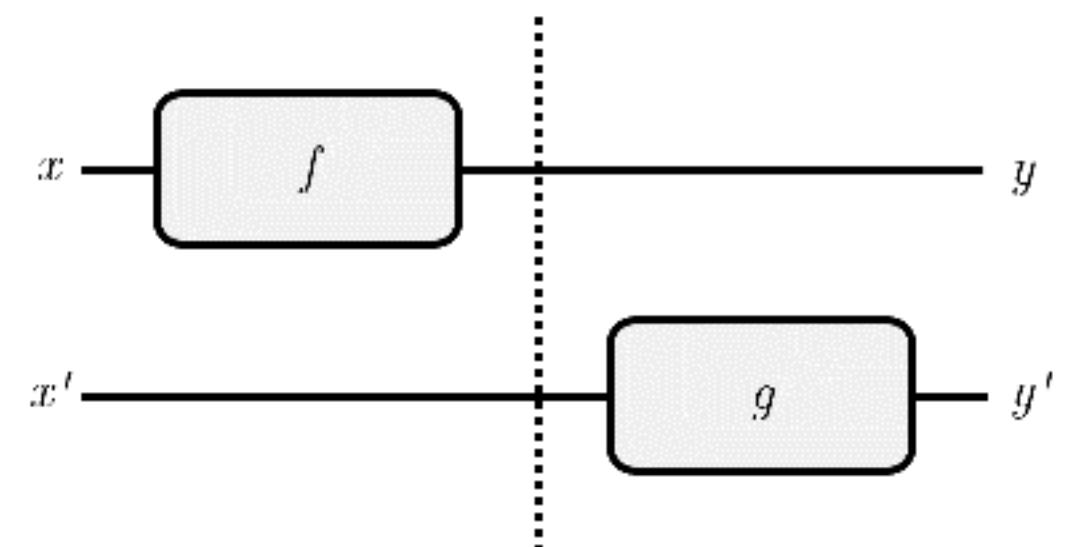
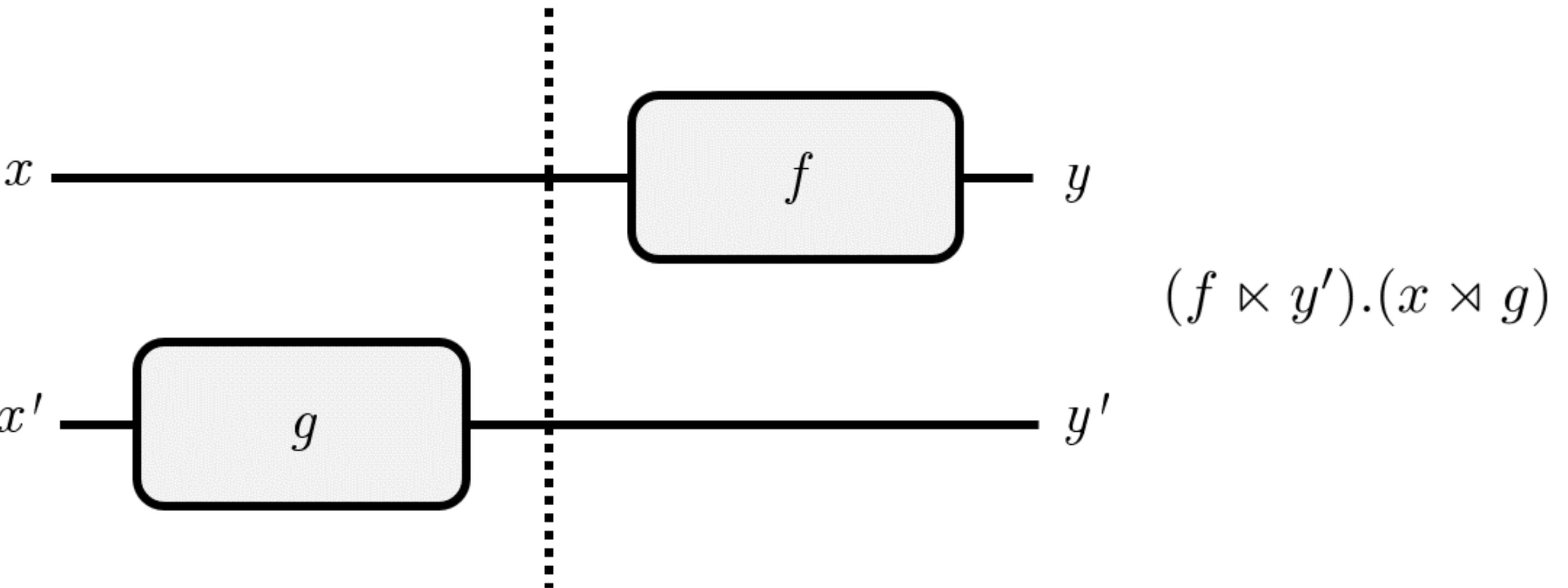
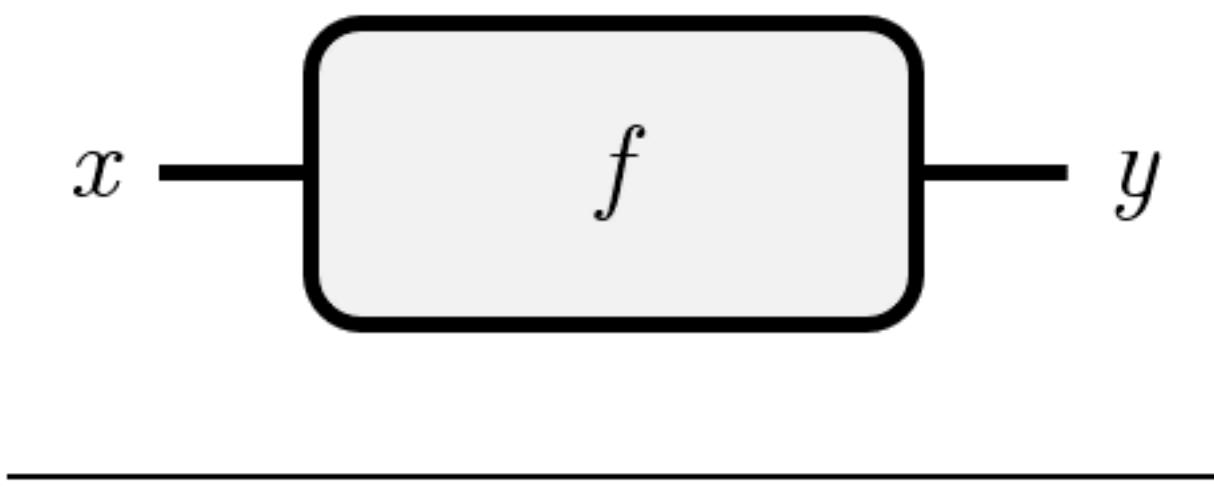
Binoidal Categories



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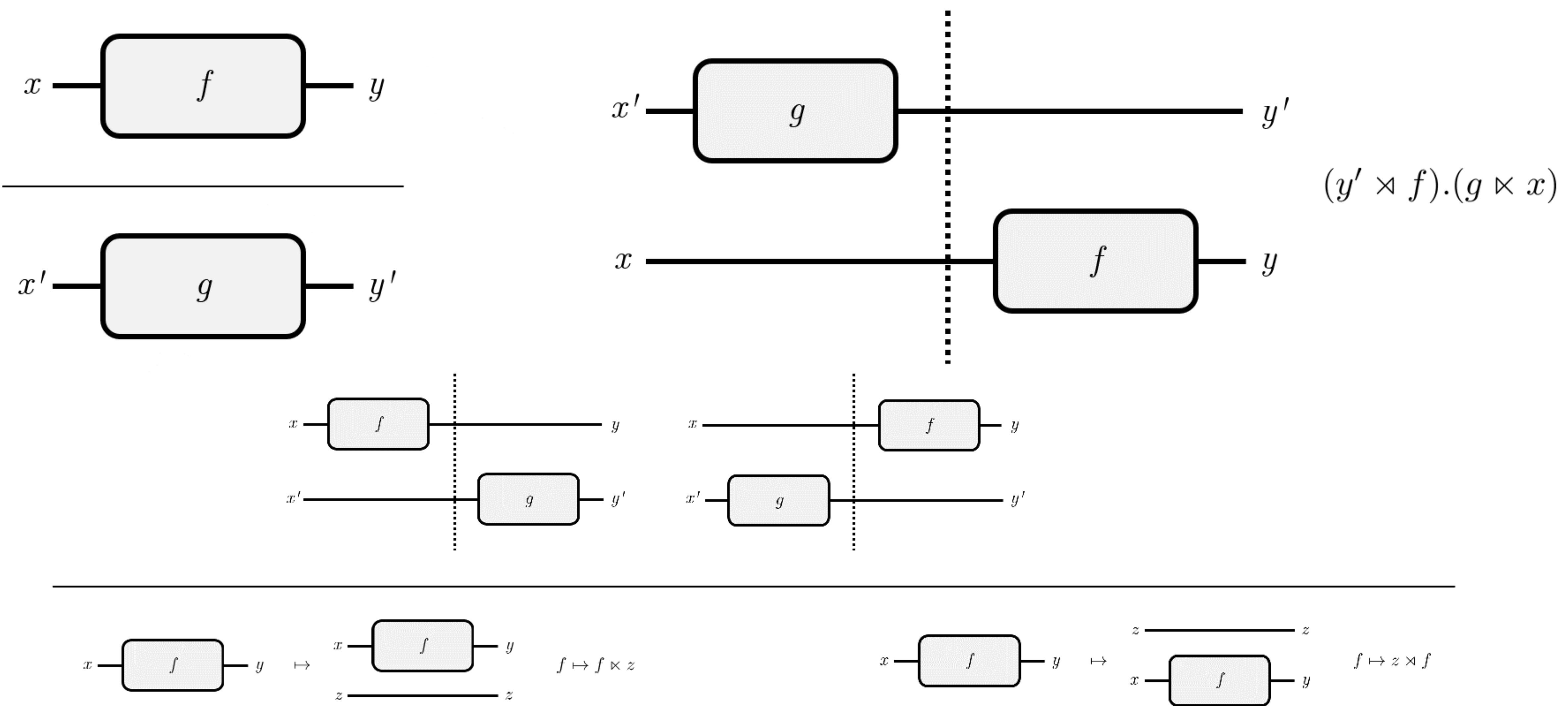
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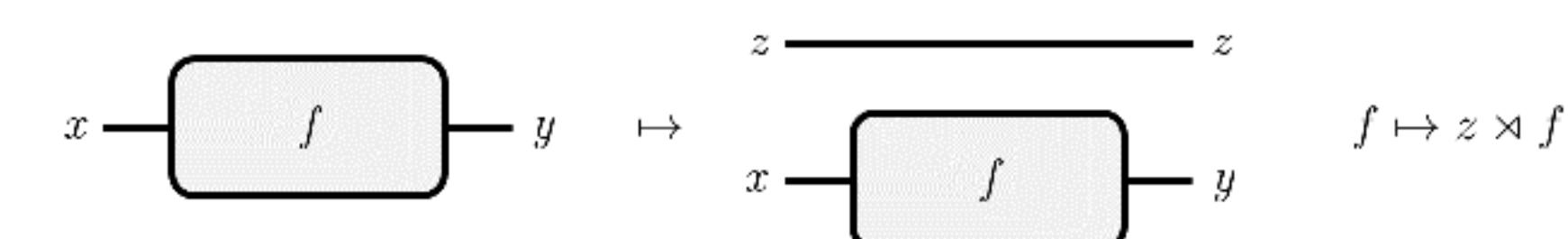
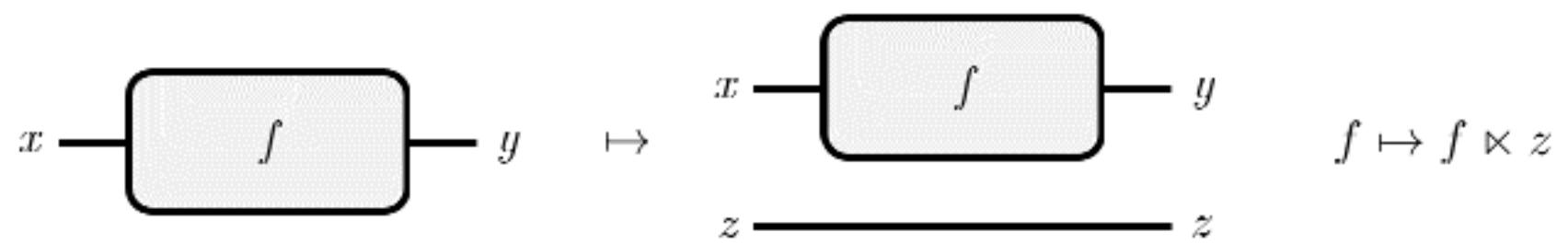
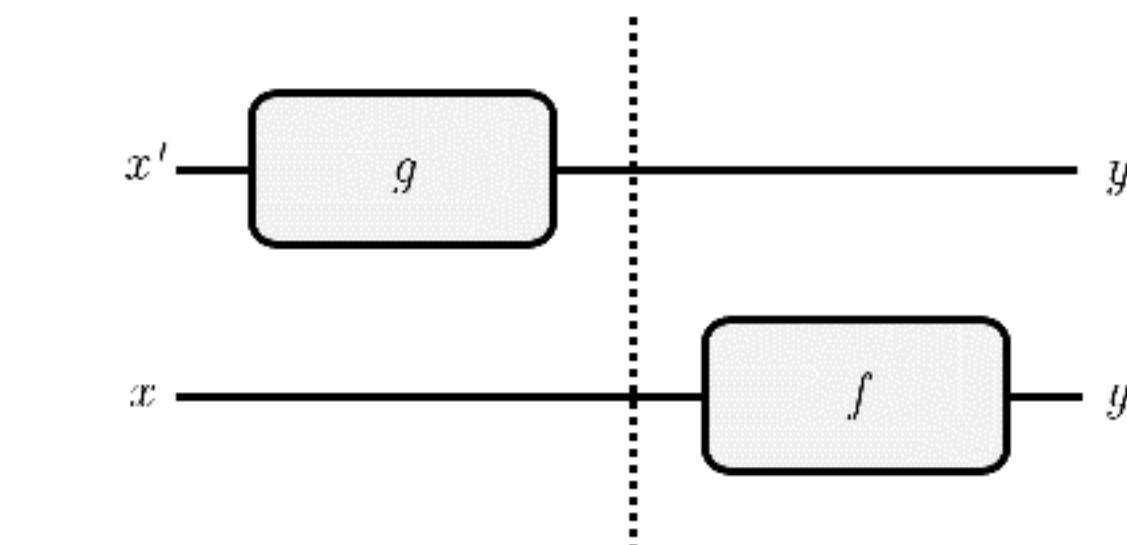
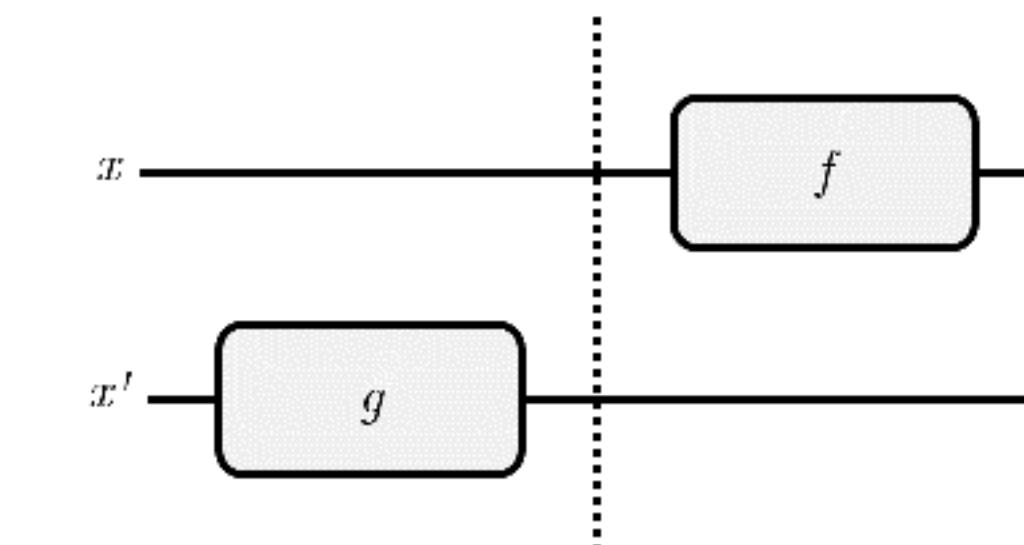
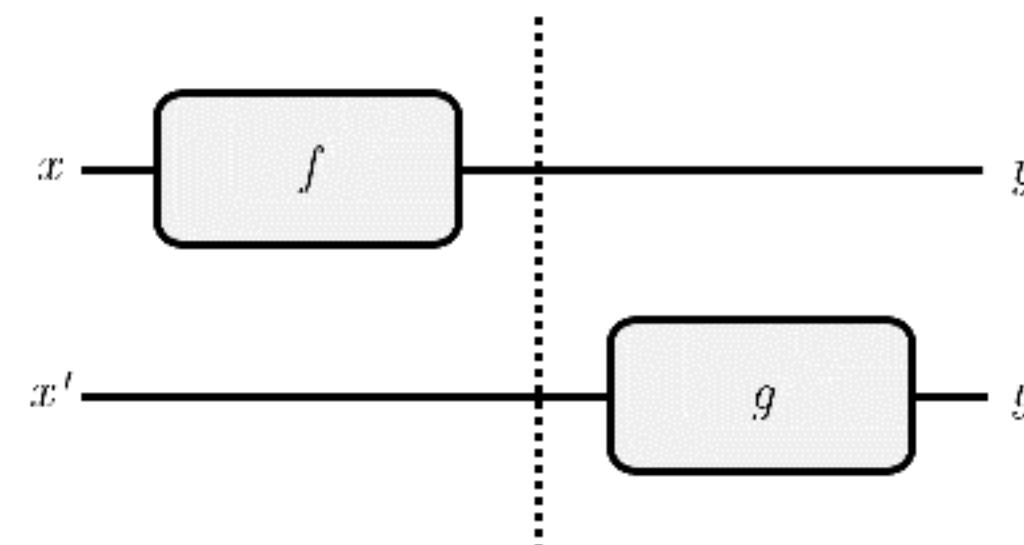
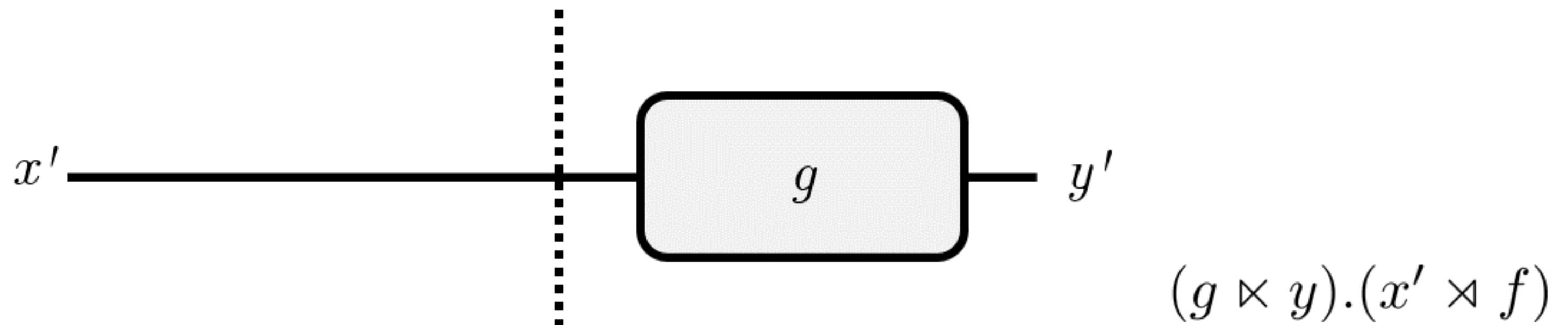
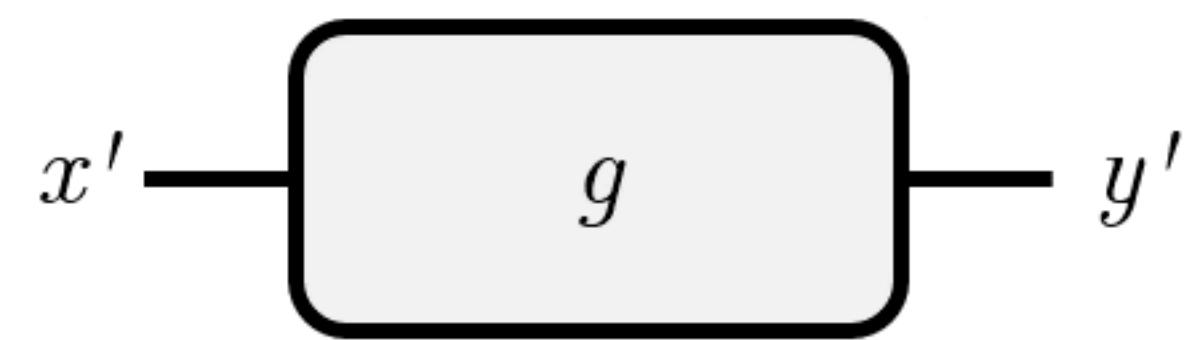
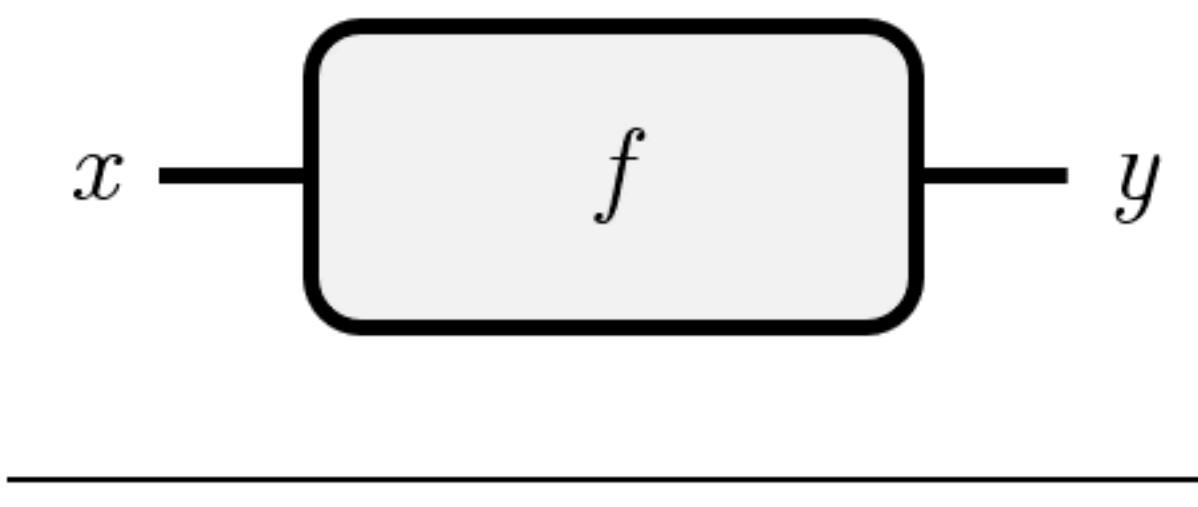
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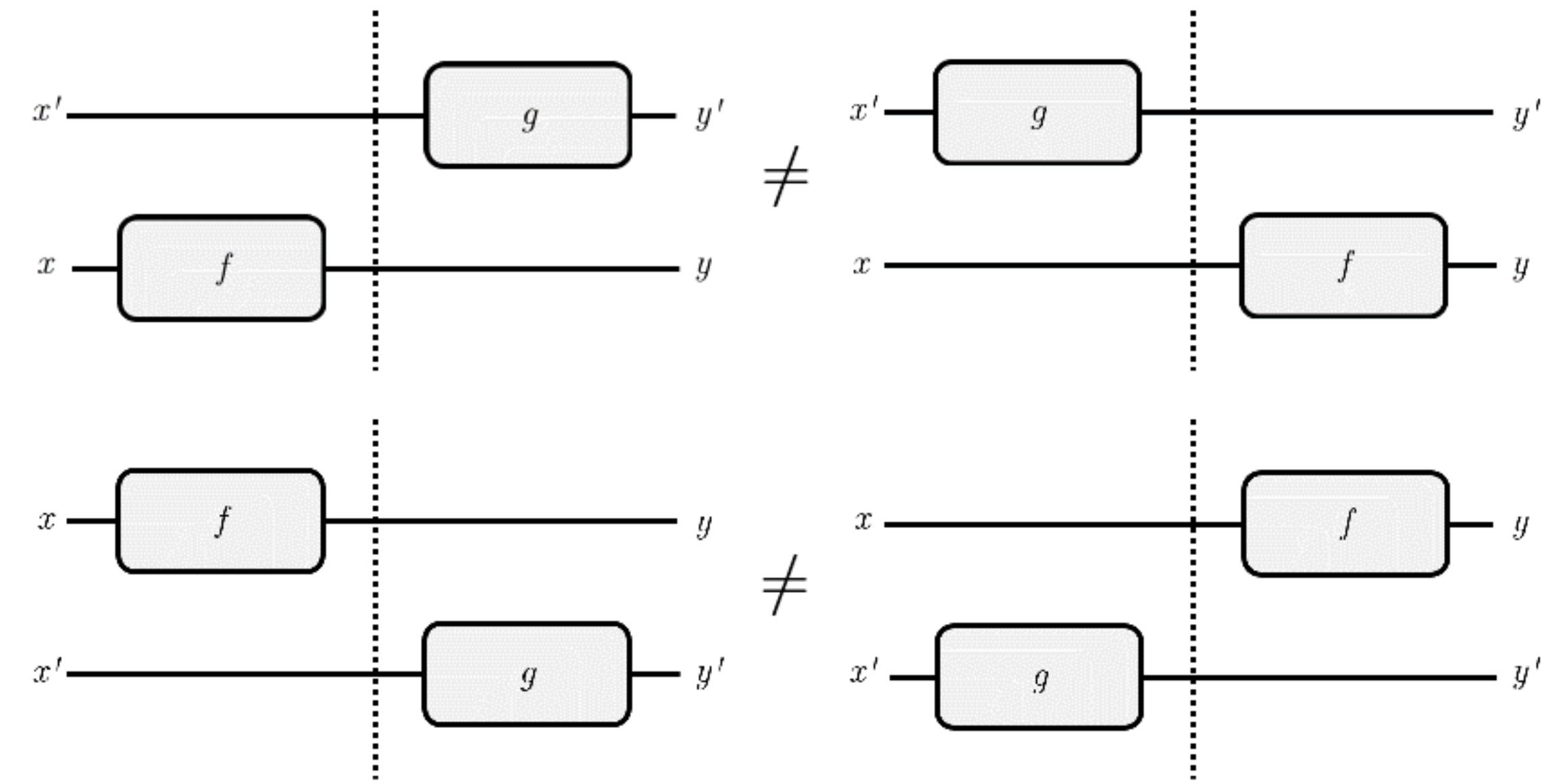
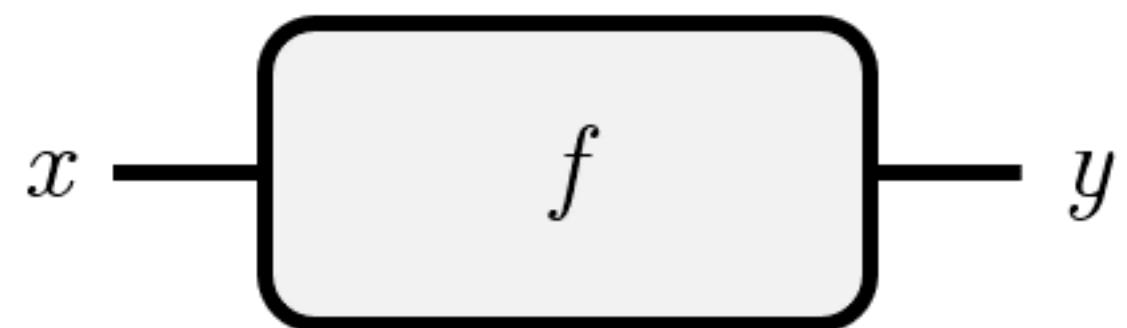
Binoidal Categories



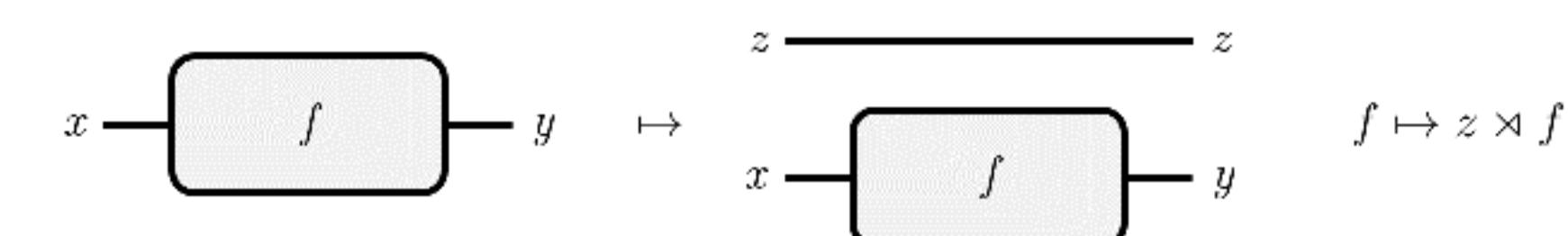
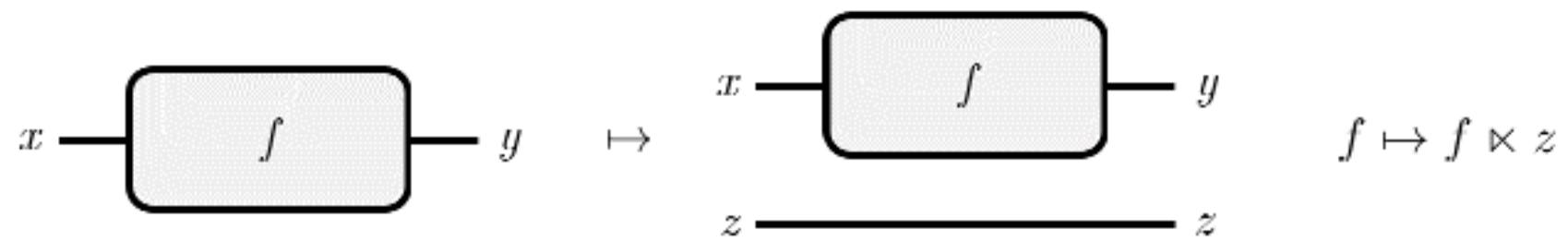
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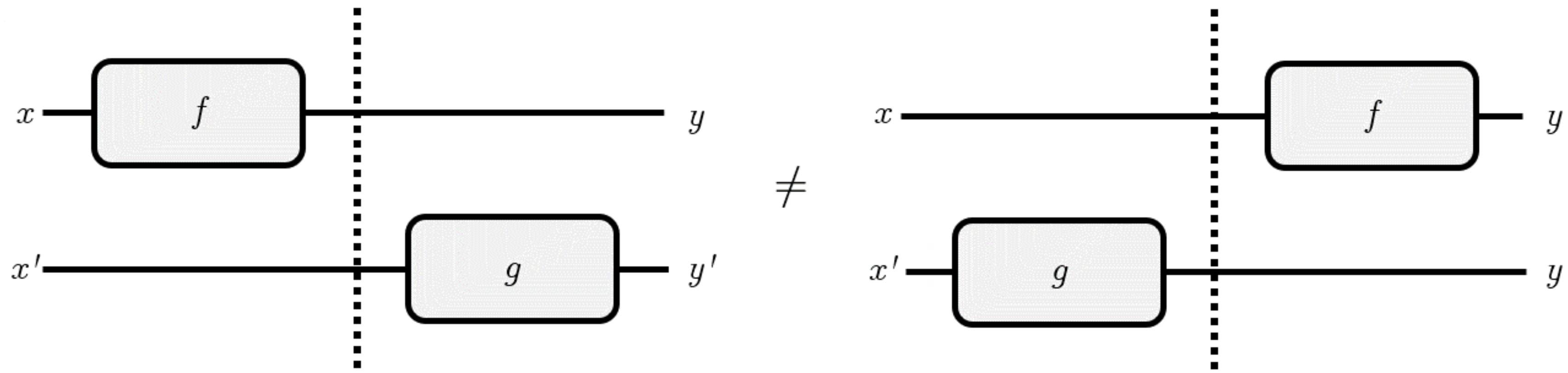
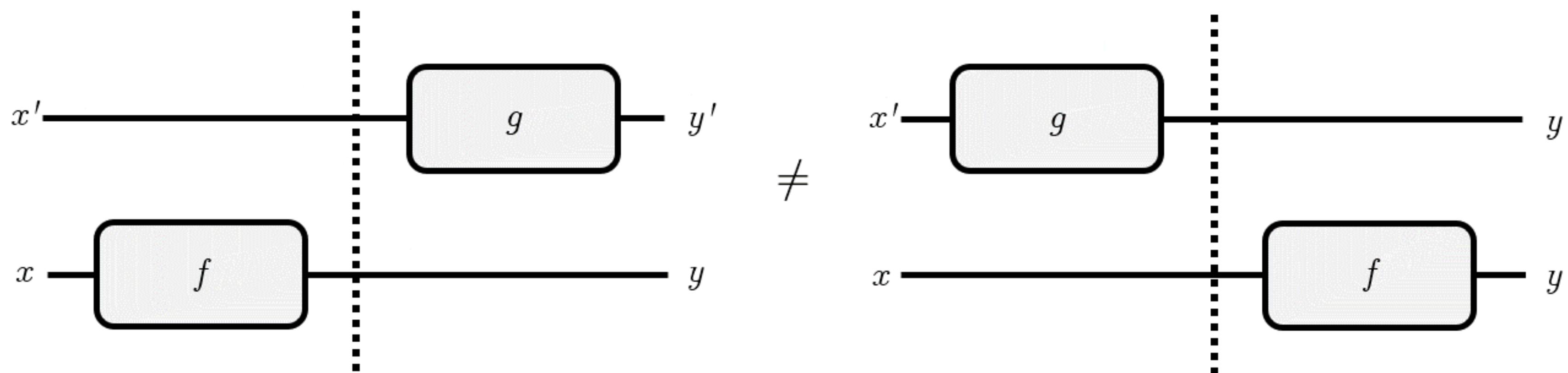
Binoidal Categories



$$(g \bowtie y).(x' \bowtie f) \neq (y' \bowtie f).(g \bowtie x) \quad (y \bowtie g).(f \bowtie x') \neq (f \bowtie y').(x \bowtie g)$$



Binoidal Categories



$$(g \ltimes y).(x' \rtimes f) \neq (y' \rtimes f).(g \ltimes x)$$

$$(y' \rtimes g).(f \ltimes x') \neq (f \ltimes y').(x \rtimes g)$$

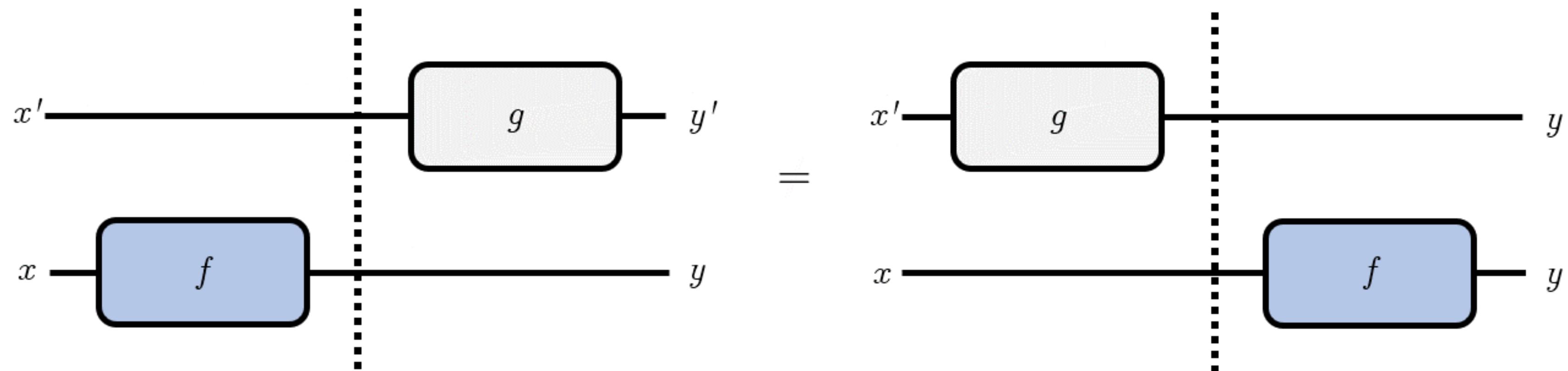
Binoidal Categories

$$\begin{array}{c}
 \begin{array}{ccc}
 x' & \xrightarrow{\hspace{1cm}} & y' \\
 & \downarrow & \\
 & g & \\
 & \downarrow & \\
 x & \xrightarrow{\hspace{1cm}} & y
 \end{array}
 & = &
 \begin{array}{ccc}
 x' & \xrightarrow{\hspace{1cm}} & y' \\
 & \downarrow & \\
 & g & \\
 & \downarrow & \\
 x & \xrightarrow{\hspace{1cm}} & y
 \end{array}
 \end{array}$$

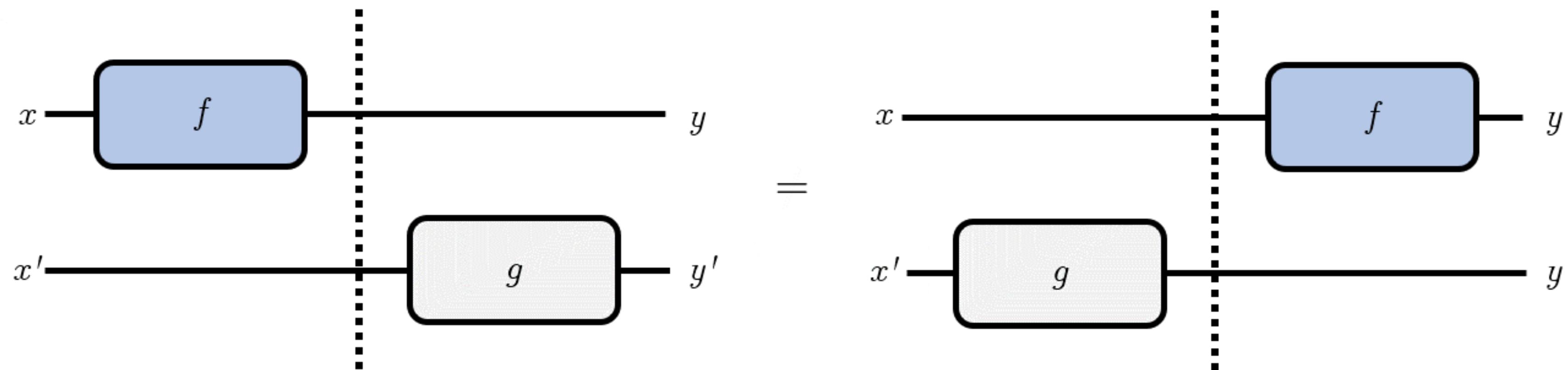
$$\begin{array}{c}
 \begin{array}{ccc}
 x & \xrightarrow{\hspace{1cm}} & y \\
 & \downarrow & \\
 & f & \\
 & \downarrow & \\
 x' & \xrightarrow{\hspace{1cm}} & y'
 \end{array}
 & = &
 \begin{array}{ccc}
 x & \xrightarrow{\hspace{1cm}} & y \\
 & \downarrow & \\
 & f & \\
 & \downarrow & \\
 x' & \xrightarrow{\hspace{1cm}} & y'
 \end{array}
 \end{array}$$

$$(g \ltimes y).(x' \rtimes f) = (y' \rtimes f).(g \ltimes x) \quad (y \rtimes g).(f \ltimes x') = (f \ltimes y').(x \rtimes g)$$

Binoidal Categories



f is *central* if this holds for all g



$$(g \ltimes y).(x' \rtimes f) = (y' \rtimes f).(g \ltimes x) \quad (y \rtimes g).(f \ltimes x') = (f \ltimes y').(x \rtimes g)$$

Binoidal Categories

$$\begin{array}{ccc}
 \begin{array}{c} x' \xrightarrow{\hspace{1cm}} g \xrightarrow{\hspace{1cm}} y' \\ x \xrightarrow{\hspace{1cm}} f \xrightarrow{\hspace{1cm}} y \end{array} & = & \begin{array}{c} x' \xrightarrow{\hspace{1cm}} g \xrightarrow{\hspace{1cm}} y' \\ x \xrightarrow{\hspace{1cm}} f \xrightarrow{\hspace{1cm}} y \end{array}
 \end{array}$$

f is *central* if this holds for all *g*

$Z(\mathbf{C})$ is the wide subcategory of central morphisms

$$\begin{array}{ccc}
 \begin{array}{c} x \xrightarrow{\hspace{1cm}} f \xrightarrow{\hspace{1cm}} y \\ x' \xrightarrow{\hspace{1cm}} g \xrightarrow{\hspace{1cm}} y' \end{array} & = & \begin{array}{c} x \xrightarrow{\hspace{1cm}} f \xrightarrow{\hspace{1cm}} y \\ x' \xrightarrow{\hspace{1cm}} g \xrightarrow{\hspace{1cm}} y' \end{array}
 \end{array}$$

$$(g \ltimes y).(x' \rtimes f) = (y' \rtimes f).(g \ltimes x) \quad (y \rtimes g).(f \ltimes x') = (f \ltimes y').(x \rtimes g)$$

Premonoidal Categories

This document is the notes for the talk

“Premonoidal Categories” by

John C. Baez and Mike Stay

on 10th May 2006 at the University of Western Ontario.

These notes were written by Mike Stay.

These notes are available online at

<http://math.ucr.edu/home/baez/premonoidal.html>

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These notes are not necessarily final or complete.

They are provided “as is” without warranty of any kind.

Premonoidal Categories

$$\begin{array}{ccc} x & \xrightarrow{\hspace{2cm}} & x \\ e & \cdots\cdots\cdots & \end{array}$$

$\rho: (x \otimes e) \rightarrow x$ a central, natural isomorphism

$$\begin{array}{ccc} e & \cdots\cdots\cdots & \\ x & \xrightarrow{\hspace{2cm}} & x \end{array}$$

$\lambda: (e \otimes x) \rightarrow x$ a central, natural isomorphism

Premonoidal Categories

$$\begin{array}{ccc} x & \xrightarrow{\hspace{2cm}} & x \\ e & \cdots\cdots\cdots & \end{array}$$

$\rho: (x \otimes e) \rightarrow x$ a central, natural isomorphism

$$\begin{array}{ccc} e & \cdots\cdots\cdots & \\ x & \xrightarrow{\hspace{2cm}} & x \end{array}$$

$\lambda: (e \otimes x) \rightarrow x$ a central, natural isomorphism

$$\begin{array}{ccc} x & \xrightarrow{\hspace{2cm}} & x \\ y & \xrightarrow{\hspace{2cm}} & y \\ z & \xrightarrow{\hspace{2cm}} & z \end{array}$$

$\alpha: (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$ a central, natural isomorphism

Premonoidal Categories

$$\begin{array}{ccc} x & \xrightarrow{\hspace{2cm}} & x \\ e & \cdots\cdots\cdots & \end{array}$$

$\rho: (x \otimes e) \rightarrow x$ a central, natural isomorphism

$$\begin{array}{ccc} e & \cdots\cdots\cdots & \\ x & \xrightarrow{\hspace{2cm}} & x \end{array}$$

$\lambda: (e \otimes x) \rightarrow x$ a central, natural isomorphism

$$\begin{array}{ccc} x & \xrightarrow{\hspace{2cm}} & x \\ y & \xrightarrow{\hspace{2cm}} & y \\ z & \xrightarrow{\hspace{2cm}} & z \end{array}$$

$\alpha: (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$ a central, natural isomorphism

satisfying the triangle and pentagon equations

(Pre)monoidal Functor

The definition of the monoidal functor

is given by the following diagram

and the commutative condition

is given by the following diagram

and the commutative condition

is given by the following diagram

and the commutative condition

is given by the following diagram

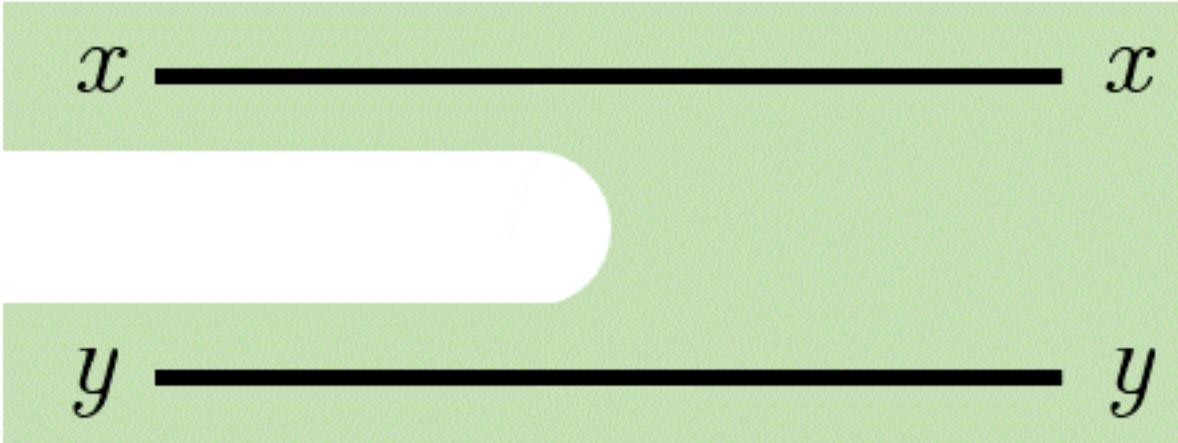
and the commutative condition

is given by the following diagram

and the commutative condition

(Pre)monoidal Functor

$$\eta: e_{\mathbf{D}} \rightarrow e \xrightarrow{\hspace{1cm}} F(e_{\mathbf{C}})$$

$$\mu: F(x) \otimes_{\mathbf{D}} F(y) \rightarrow F(x \otimes_{\mathbf{C}} y)$$


(Pre)monoidal Functor

$$\eta: e_{\mathbf{D}} \quad e \xrightarrow{\hspace{10em}} \quad F(e_{\mathbf{C}})$$

$$\mu: F(x) \otimes_{\mathbf{D}} F(y)$$

$$x \xrightarrow{\hspace{10em}} x \\ F(x \otimes_{\mathbf{C}} y)$$

y $\xrightarrow{\hspace{10em}} y$

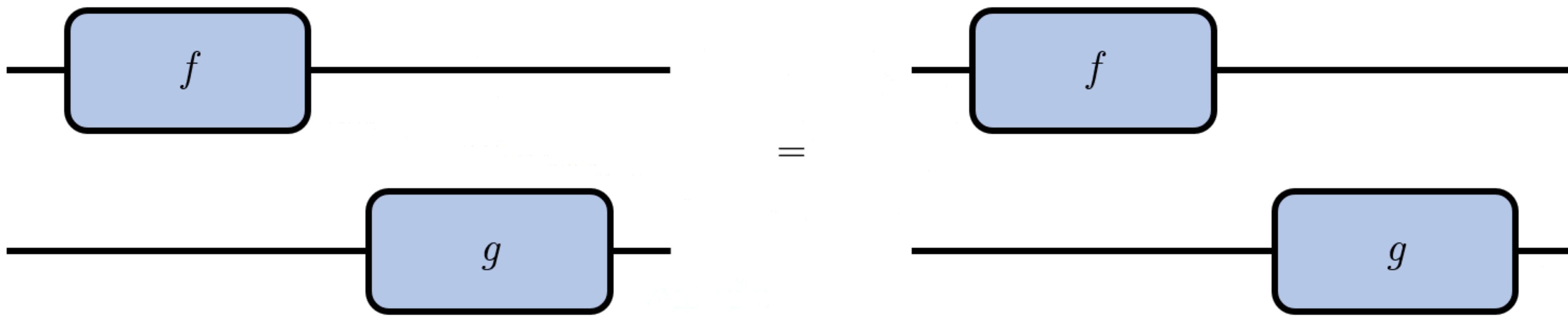
$$e_{\mathbf{D}} \otimes_{\mathbf{D}} F(x) \quad x \xrightarrow{\hspace{10em}} x \quad = \quad x \xrightarrow{\hspace{10em}} x \quad F(x) \quad \text{and symmetric}$$

$e \xrightarrow{\hspace{10em}}$ $e \xrightarrow{\hspace{10em}}$

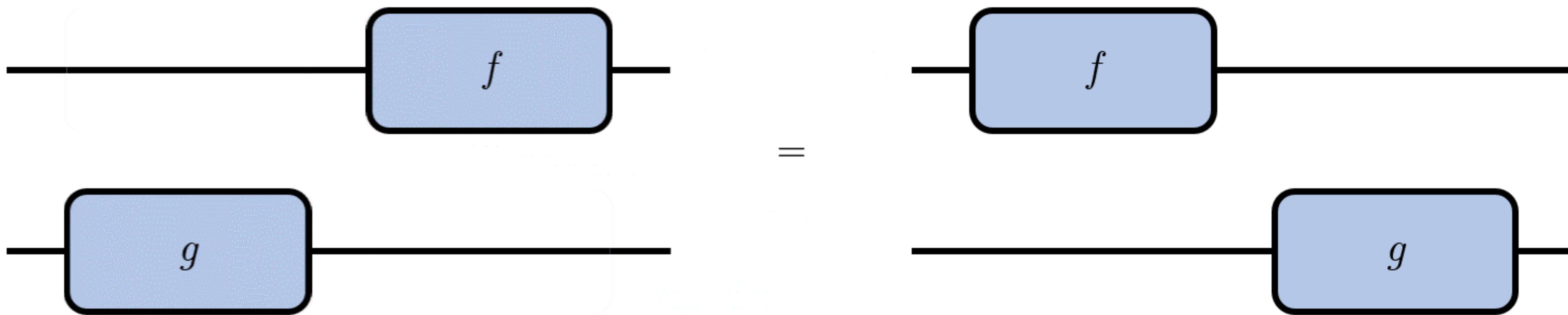
$$(F(x) \otimes_{\mathbf{D}} F(y)) \otimes_{\mathbf{D}} F(z) \quad x \xrightarrow{\hspace{10em}} x \\ y \xrightarrow{\hspace{10em}} y \\ z \xrightarrow{\hspace{10em}} z \quad = \quad y \xrightarrow{\hspace{10em}} y \\ z \xrightarrow{\hspace{10em}} z \quad F(x \otimes_{\mathbf{C}} (y \otimes_{\mathbf{C}} z))$$

Freyd Categories

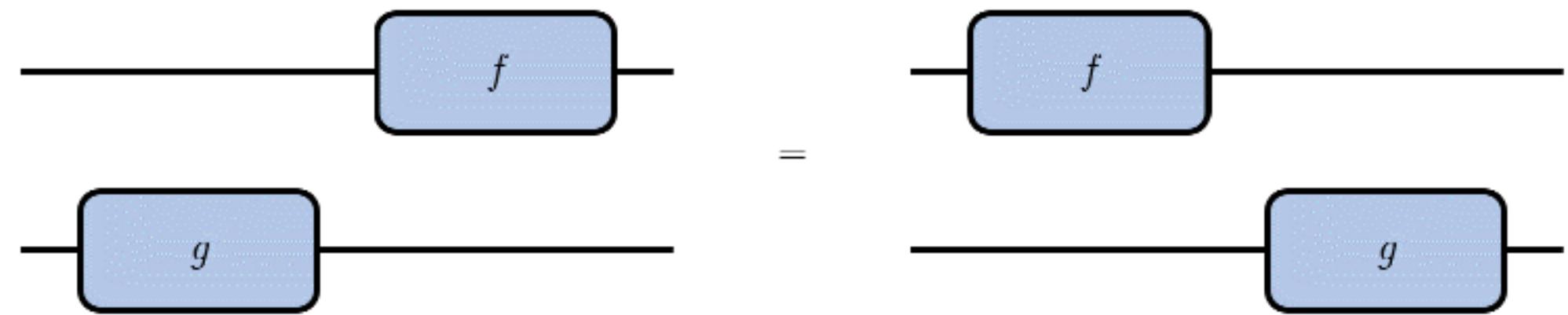
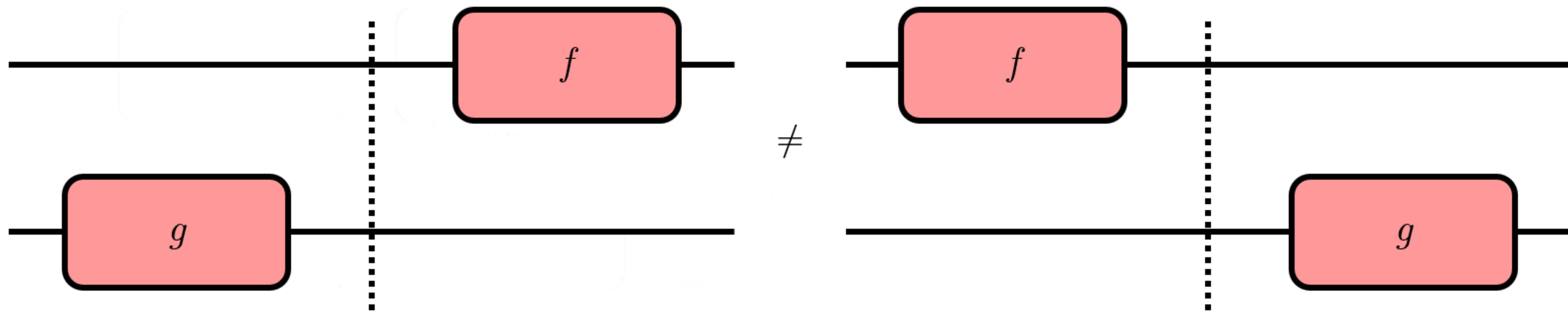
Freyd Categories



Freyd Categories

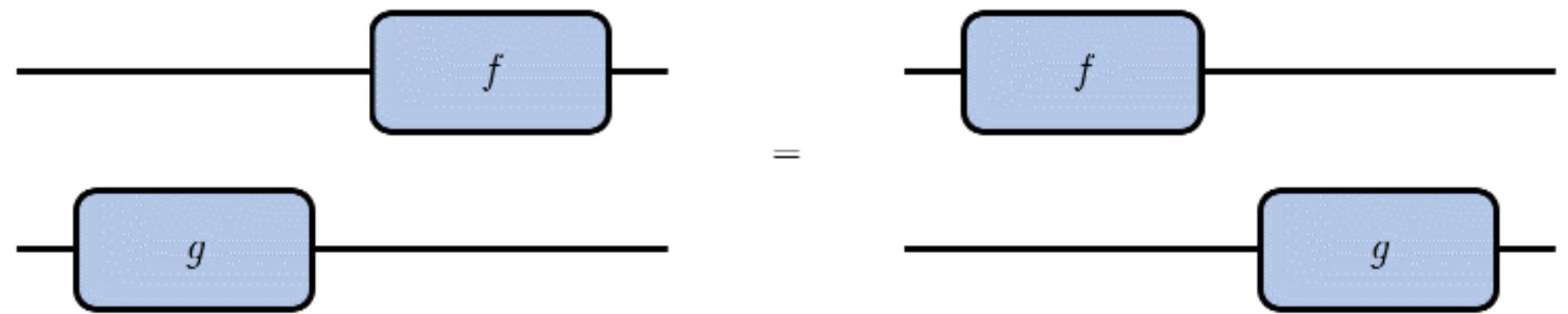
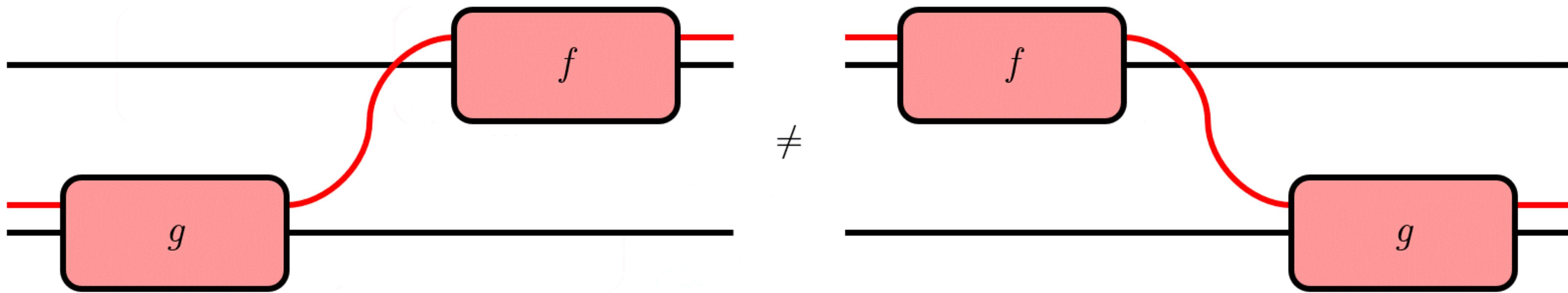


Freyd Categories



monoidal \mathbf{M}

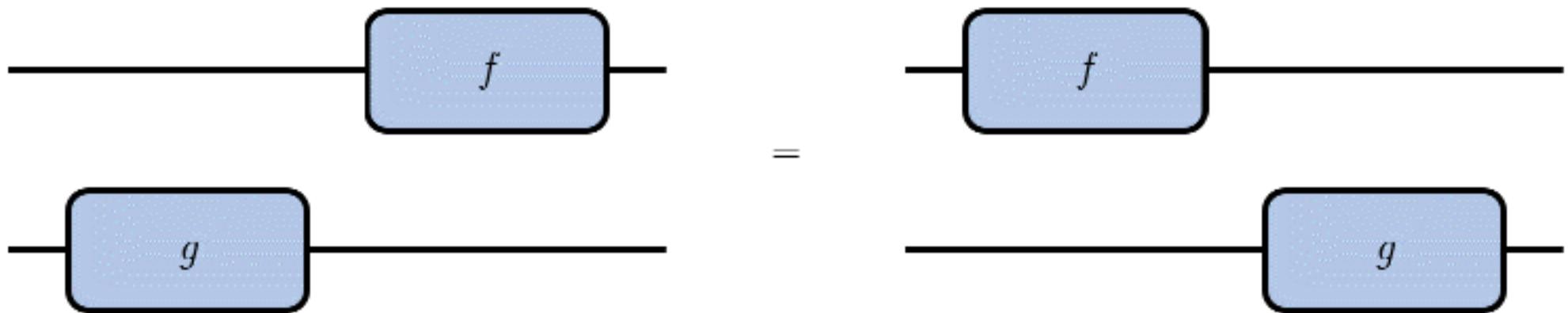
Freyd Categories



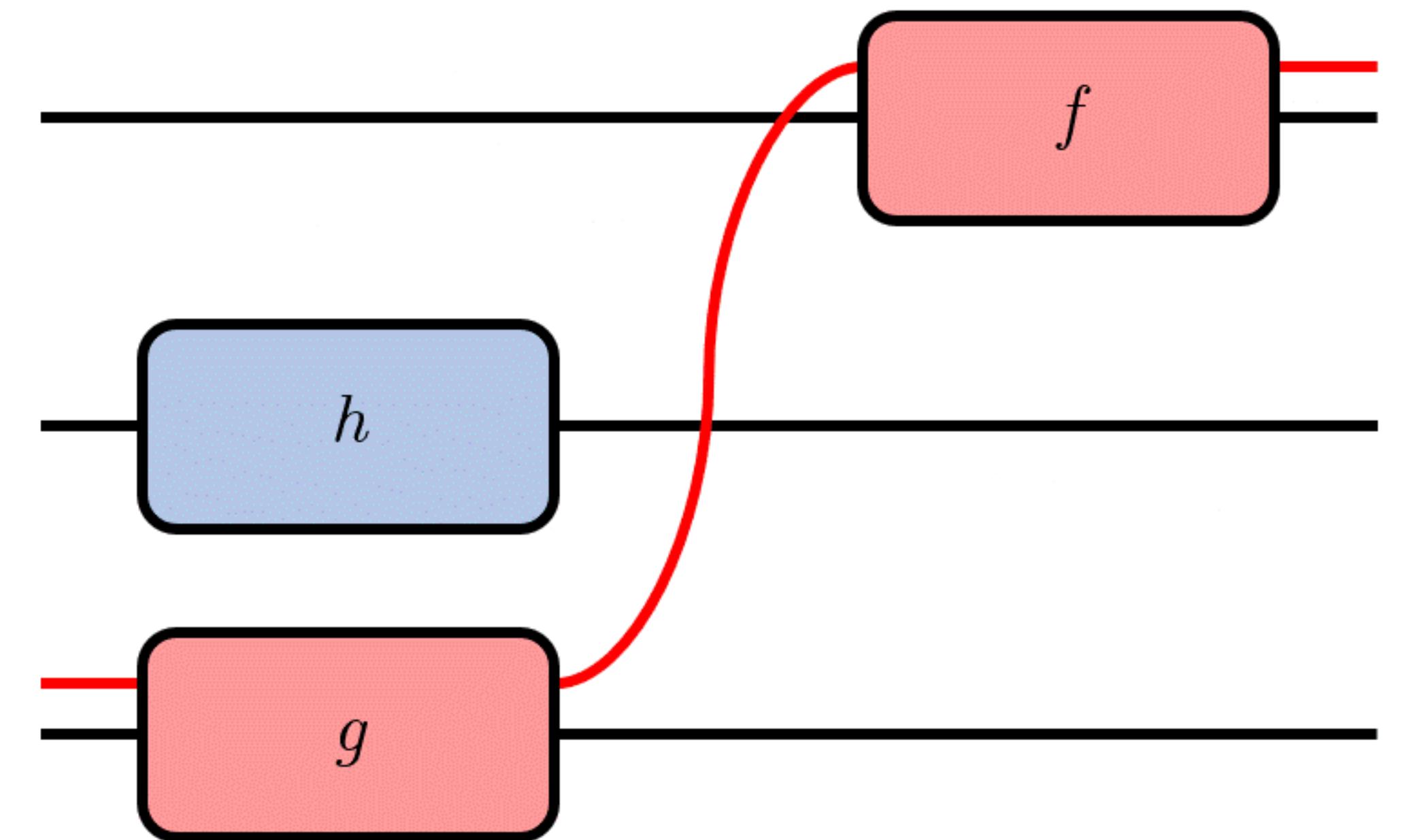
monoidal \mathbf{M}

Freyd Categories

an identity-on-objects strict premonoidal functor $J: \mathbf{M} \rightarrow \mathbf{C}$
whose image lies in $Z(\mathbf{C})$



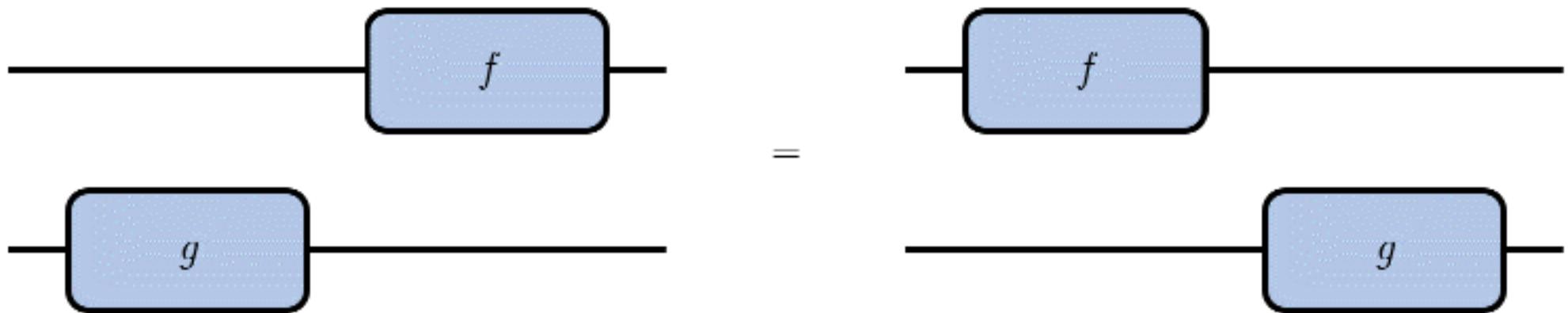
monoidal \mathbf{M}



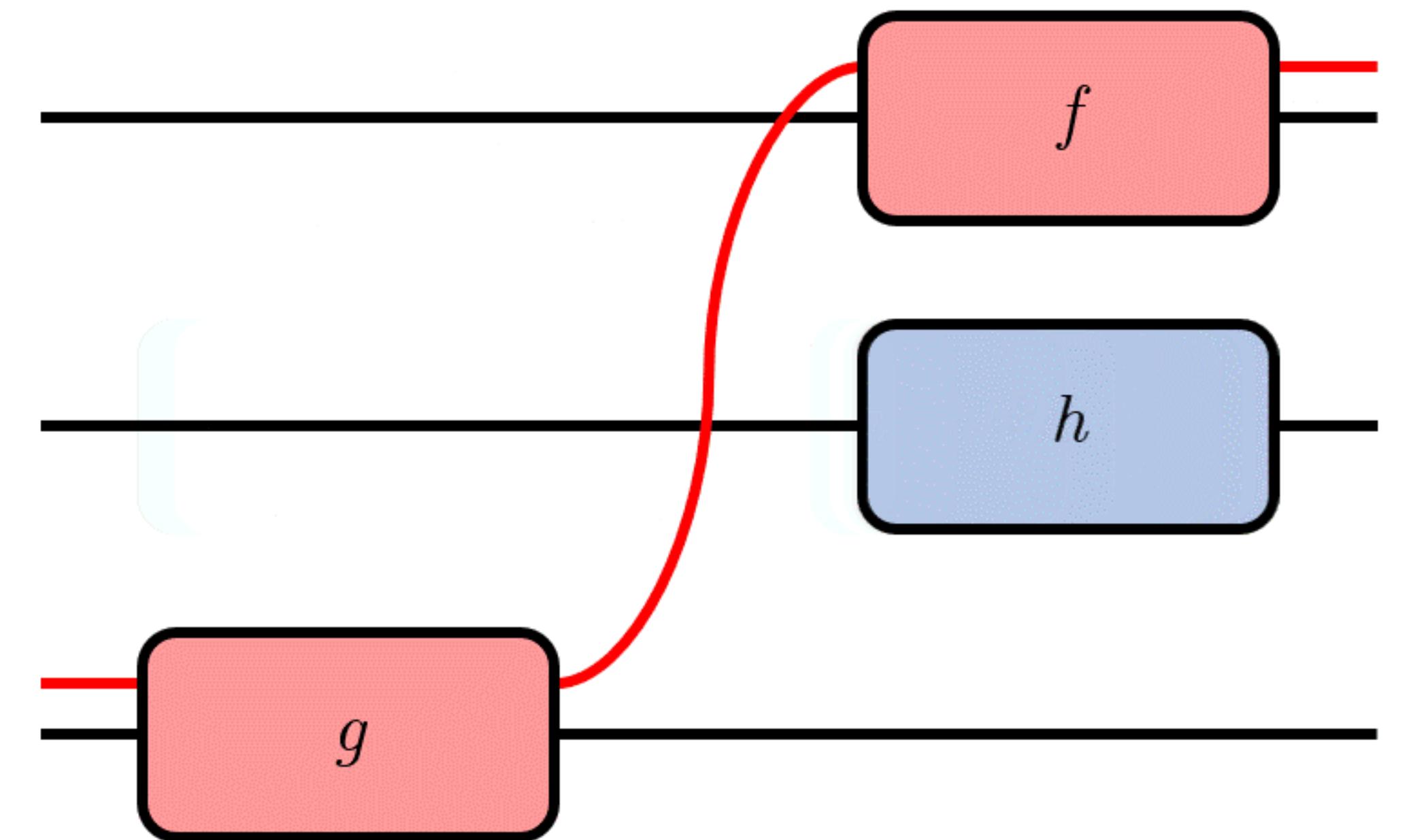
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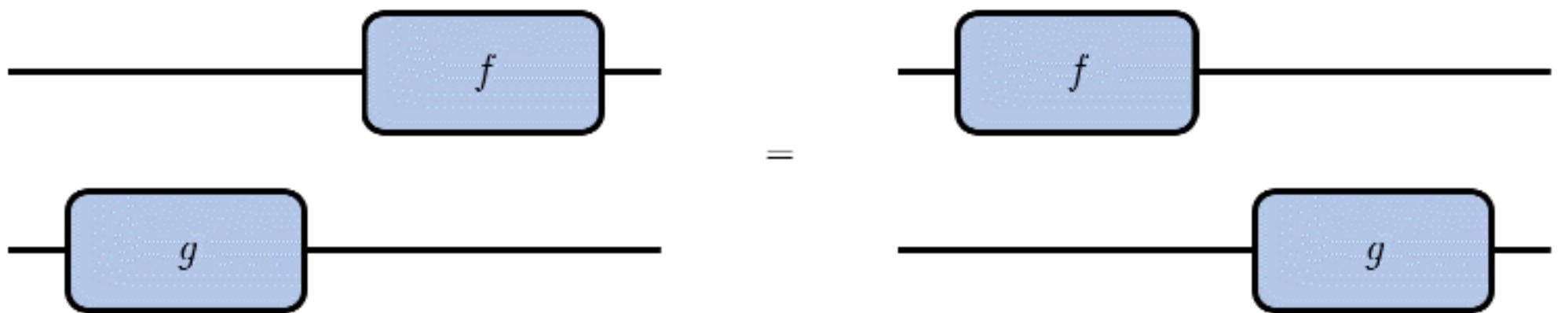


premonoidal \mathbf{C}

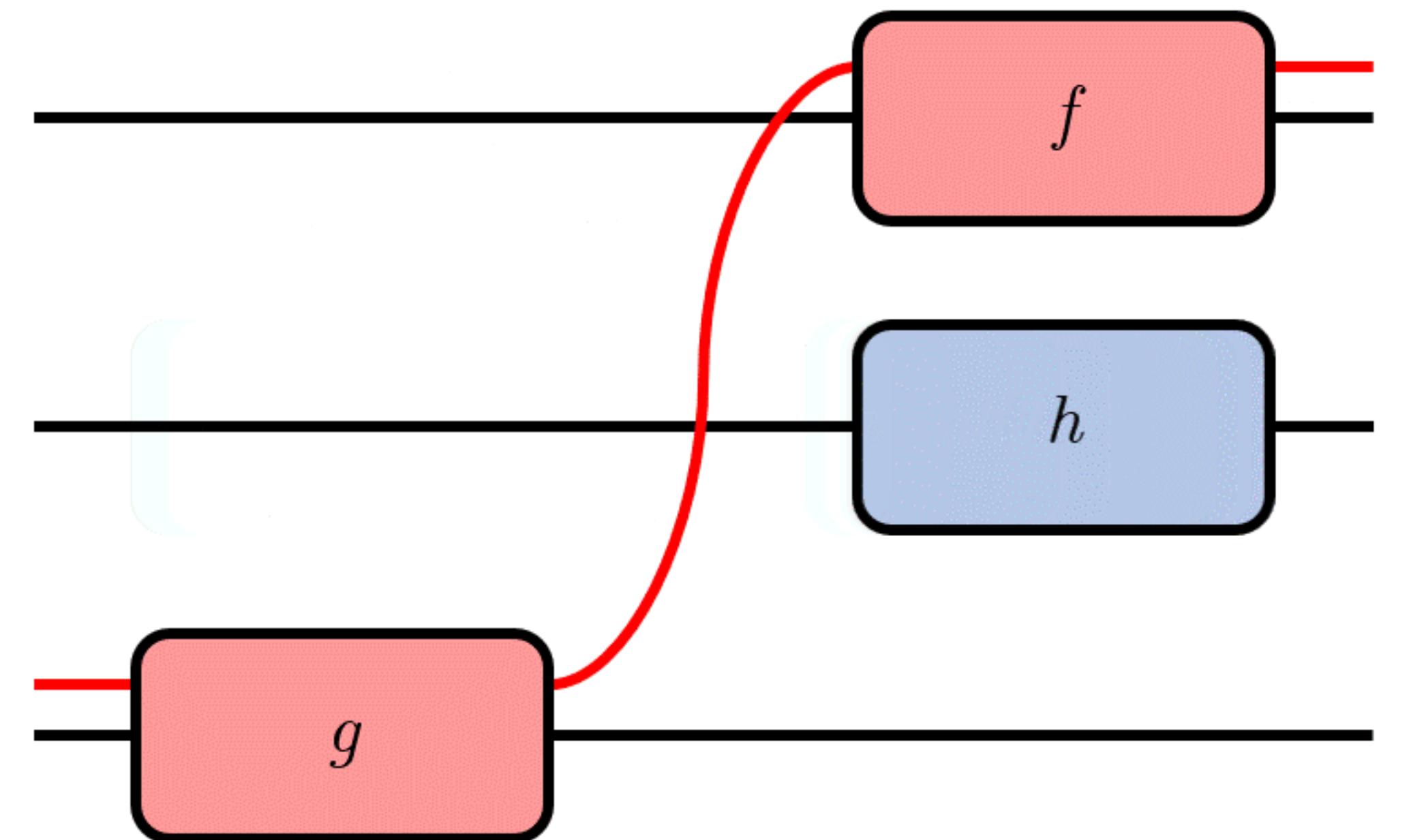
Freyd Categories

an identity-on-objects strict premonoidal functor $J: \mathbf{M} \rightarrow \mathbf{C}$
 whose image lies in $Z(\mathbf{C})$

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{F_0} & \mathbf{M}' \\ J \downarrow & & \downarrow J' \\ \mathbf{C} & \xrightarrow{F_1} & \mathbf{C}' \end{array} \quad \begin{array}{l} F_0 \text{ strong monoidal} \\ F_1 \text{ strong premonoidal} \end{array}$$



monoidal \mathbf{M}



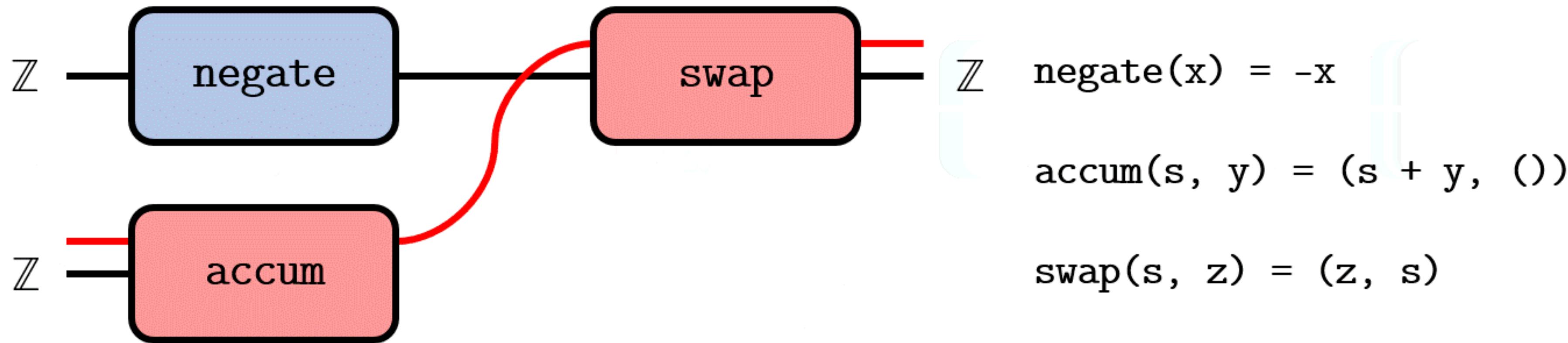
premonoidal \mathbf{C}

Freyd Category Example



Freyd Category Example

$$\mathbf{M} = (\mathbf{Set}, \times, 1) \quad \mathbf{C}(a, b) := \mathbf{Set}(\mathbb{Z} \times a, \mathbb{Z} \times b) \quad f \ltimes c := f \times \text{id}_c \quad J(f) := \text{id}_{\mathbb{Z}} \times f$$



Duoidal Categories

• \mathcal{C} is a duoidal category if it has two dualities: one for objects and one for morphisms.

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Duoidal Categories

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$(\mathbf{V}, *, J)$, parallel

(\mathbf{V}, \circ, I) , sequential



Duoidal Categories

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$$\zeta_{A,B,C,D}: (A \circ B) * (C \circ D) \rightarrow (A * C) \circ (B * D)$$

$$\Delta: J \rightarrow J \circ J \quad \nabla: I * I \rightarrow I \quad \epsilon: J \rightarrow I$$

$$*\frac{A \circ B}{C \circ D} \xrightarrow{\zeta} \begin{array}{c} \textcircled{O} \\ A \vdash B \\ \ast \\ C \vdash D \end{array} \qquad J \xrightarrow{\Delta} \begin{array}{c} \textcircled{O} \\ J \vdash J \end{array} \quad *\frac{I}{I} \xrightarrow{\nabla} I \qquad J \xrightarrow{\epsilon} I$$

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(I, ∇, ϵ) is a monoid in $(\mathbf{V}, *, J)$ (J, Δ, ϵ) is a comonoid in (\mathbf{V}, \circ, I)

Duoidal Categories

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$$*\frac{A \circ B}{C \circ D} \xrightarrow{\zeta} \begin{array}{c} \textcircled{O} \\ A \\ * \\ C \end{array} \begin{array}{c} B \\ * \\ D \end{array} \quad J \xrightarrow{\Delta} \begin{array}{c} \textcircled{O} \\ J \\ | \\ J \end{array} \quad * \frac{I}{I} \xrightarrow{\nabla} I \quad J \xrightarrow{\epsilon} I$$

$$\begin{array}{c} I \quad I \\ \hline A \quad B \end{array} \xleftarrow{A \quad B} \begin{array}{c} A \quad B \\ \hline I \quad I \end{array} \quad \begin{array}{c} I \quad I \\ \hline A \quad B \end{array} \xrightarrow{A \quad B} \begin{array}{c} A \quad B \\ \hline I \quad I \end{array}$$

$$\begin{array}{c} J \quad A \\ \hline J \quad B \end{array} \xleftarrow{A \quad B} \begin{array}{c} A \\ \hline B \end{array} \quad \begin{array}{c} J \quad A \\ \hline J \quad B \end{array} \xrightarrow{A \quad B} \begin{array}{c} A \quad J \\ \hline B \quad J \end{array}$$

$$*\frac{A \circ B}{C \circ D} \xrightarrow{\zeta} \begin{array}{c} \circ \\ A \\ * \\ C \end{array} \begin{array}{c} B \\ * \\ D \end{array} \qquad J \xrightarrow{\Delta} \begin{array}{c} \circ \\ J \\ | \\ J \end{array} \qquad *\frac{I}{I} \xrightarrow{\nabla} I \qquad J \xrightarrow{\epsilon} I$$

$$\begin{array}{ccccc} I & I & & A & B \\ \hline A & B & \leftarrow A & B & \rightarrow \frac{A}{I} \frac{B}{I} \\ \downarrow & & & \downarrow & \\ I & I & \rightarrow A & B & \leftarrow A & B \\ & & A & I & & I \\ A & B & & & & I \end{array}$$

$$\begin{array}{ccccc} J & A & & A & J \\ \hline J & B & \leftarrow & B & \rightarrow \frac{A}{B} \frac{J}{J} \\ \downarrow & & & \downarrow & \\ J & A & \rightarrow & A & \leftarrow A & J \\ & & J & B & & B & J \end{array}$$

$$\begin{array}{ccccc} I & I & & I & I \\ \leftarrow & \rightarrow & & \leftarrow & \rightarrow \\ \downarrow & & & \downarrow & \\ J & I & \rightarrow & I & \leftarrow I & J \end{array}$$

$$\begin{array}{ccccc} J & & & I & \\ \hline I & & \leftarrow J & \rightarrow & \frac{I}{J} \\ \downarrow & & & & \downarrow \\ J & & \rightarrow J & \leftarrow & \frac{J}{J} \end{array}$$

Duoidal Categories Examples

• \mathbf{Set} (with \otimes and \wedge)

• \mathbf{Grpd} (with \otimes and \wedge)

• \mathbf{CAlg} (with \otimes and \wedge)

Duoidal Categories Examples

Any braided monoidal category is duoidal with ζ being the middle-four interchange $x \otimes y \otimes z \otimes w \rightarrow x \otimes z \otimes y \otimes w$.



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If $(\mathbf{V}, *, J, \circ, I)$ is duoidal, so is $(\mathbf{V}^{\text{op}}, \circ, I, *, J)$, with opposite structure maps.

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For monoidal (\mathbf{V}, \otimes, I) with products, $(\mathbf{V}, \otimes, I, \times, 1)$ is duoidal with $\zeta = \langle \pi_1 \otimes \pi_1, \pi_2 \otimes \pi_2 \rangle$. Similarly, $(\mathbf{V}, +, 0, \otimes, I)$ is duoidal.

Category of Labelled Sets

Set

Function

Object

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Set

Function

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$$A$$

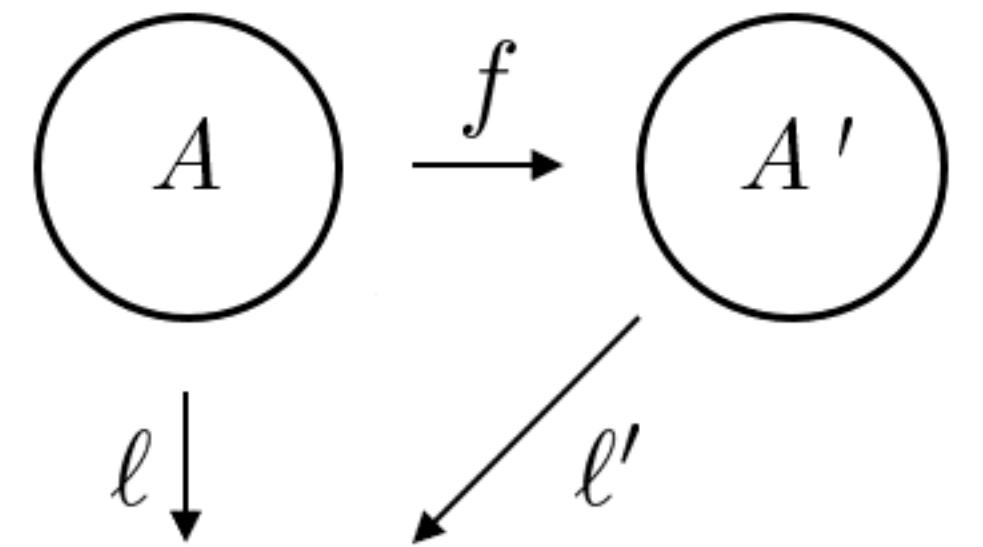
$$\ell \downarrow$$

$$\boxed{\mathcal{P}_f(R)}$$

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$$(A \times A') \xrightarrow{\ell \cup \ell'} \boxed{\mathcal{P}_f(R)}$$

$$(\ell \cup \ell')(a, a') := \ell(a) \cup \ell'(a')$$

$$A \xrightarrow{f} A' \\ \ell \downarrow \quad \searrow \ell'$$

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(Label, ||, cst_∅, ∪, cst_∅) is duoidal

Category of Distinguished Subsets

1. $\{ \}$

2. $\{ \cdot \}$

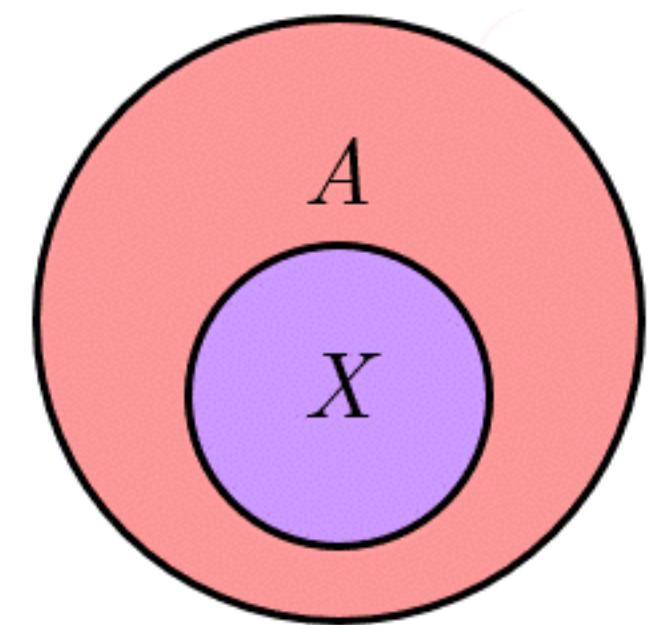
3. $\{ \cdot, \cdot \}$

4. $\{ \cdot, \cdot, \cdot \}$

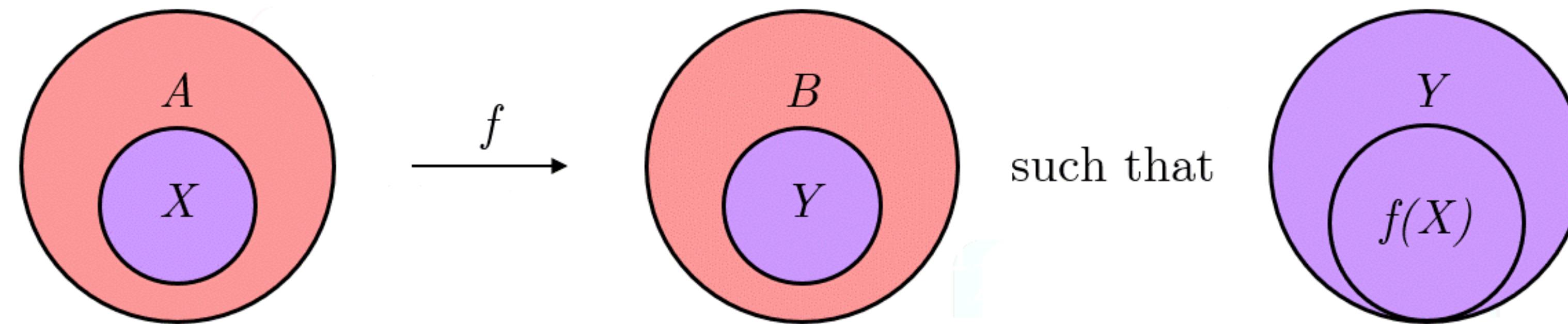
5. $\{ \cdot, \cdot, \cdot, \cdot \}$

6. $\{ \cdot, \cdot, \cdot, \cdot, \cdot \}$

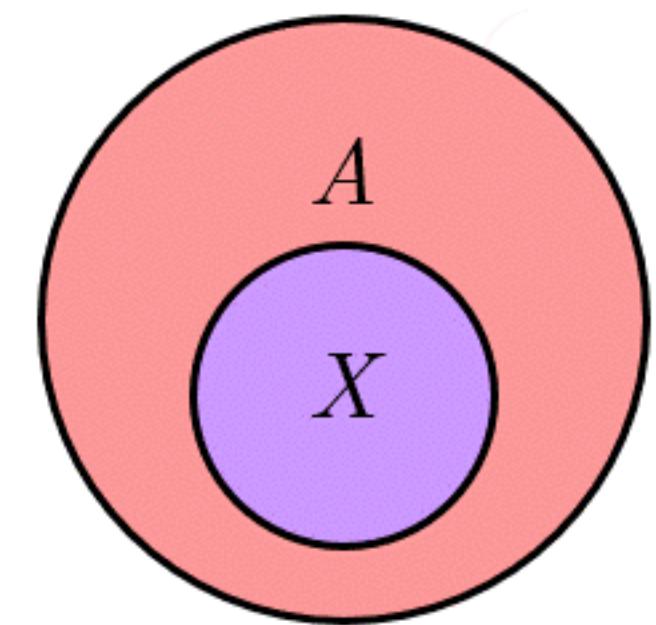
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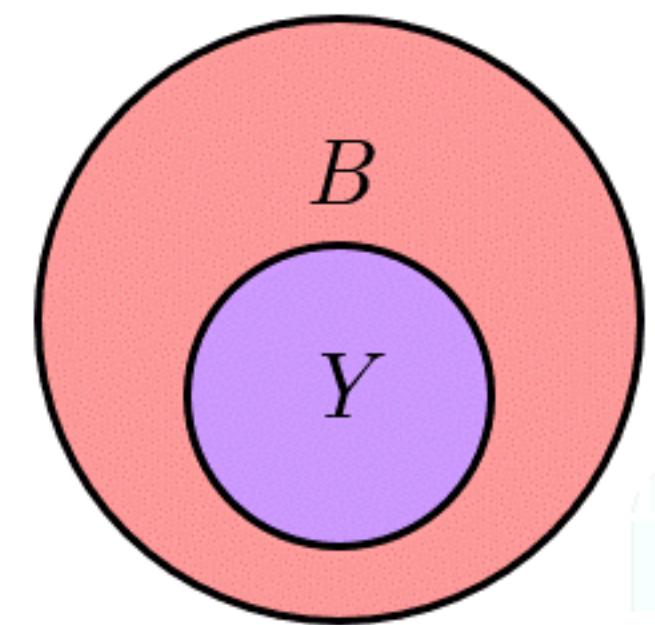
Category of Distinguished Subsets



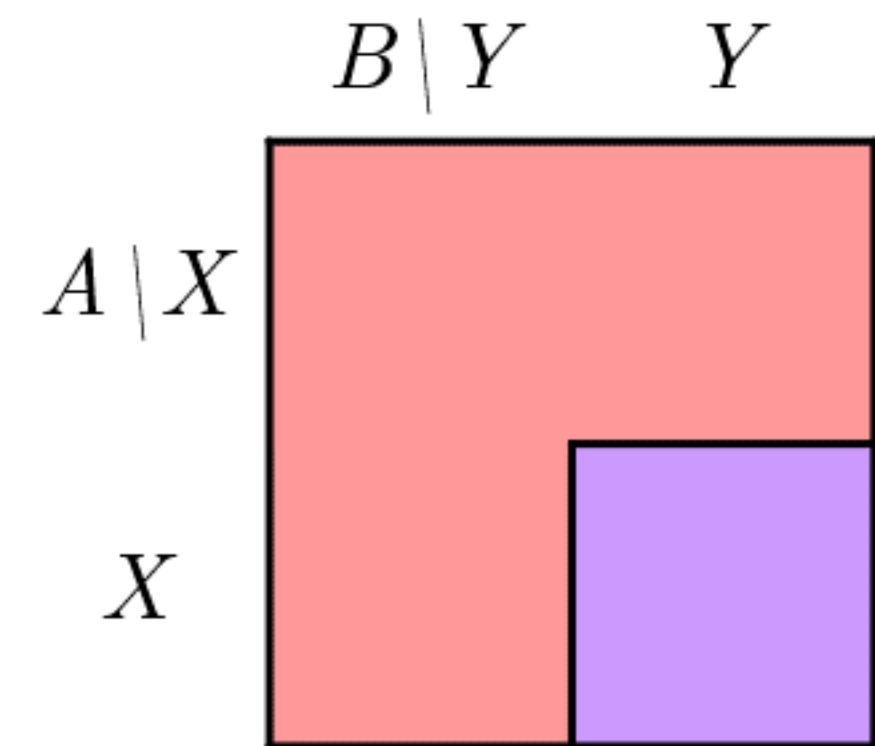
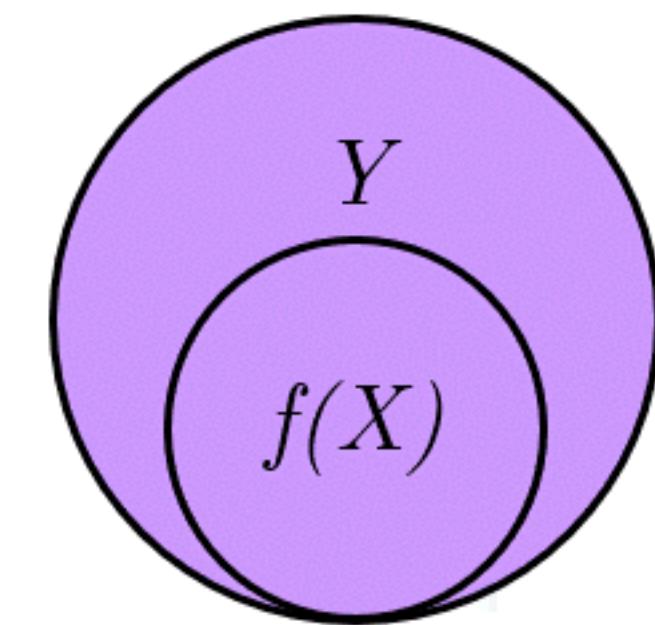
Category of Distinguished Subsets



f

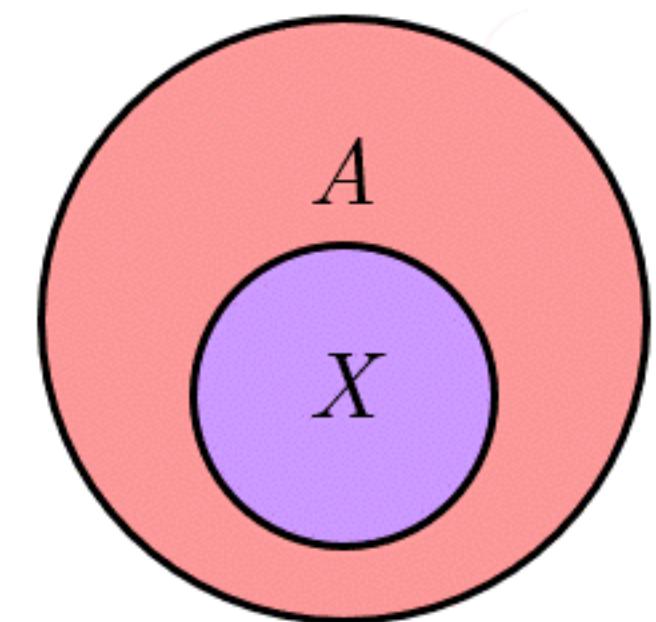


such that

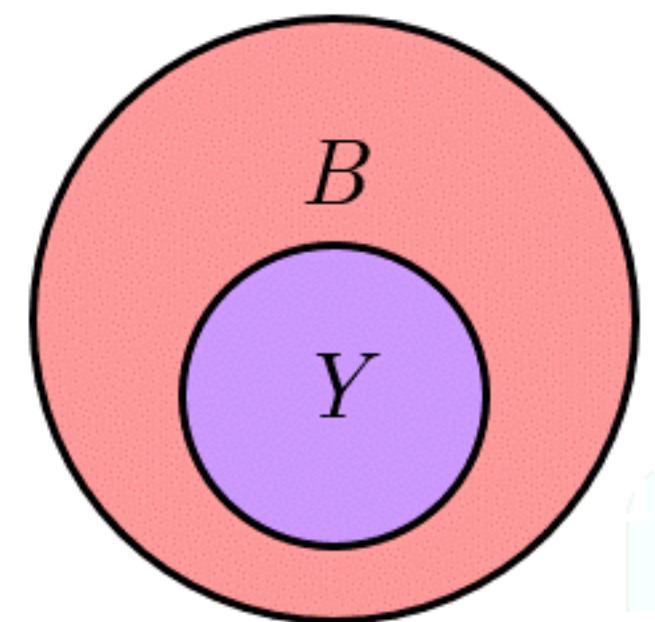


$$(X, A) \times (Y, B) := (X \times Y, A \times B)$$

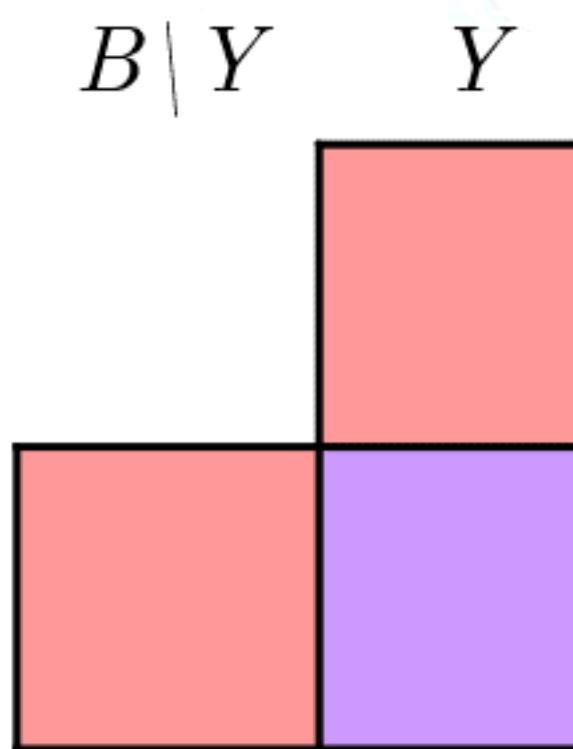
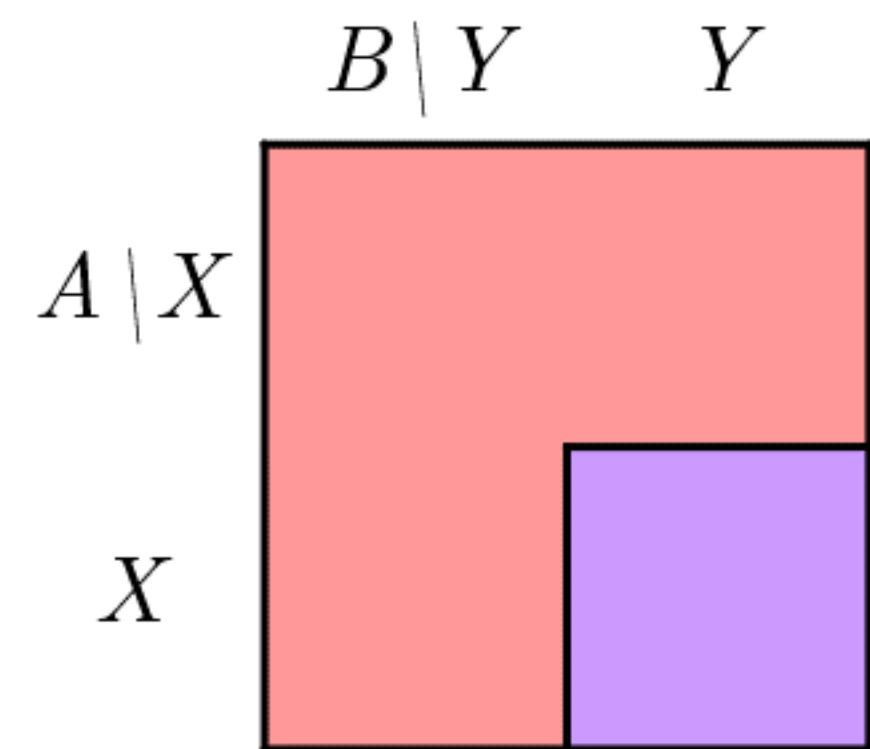
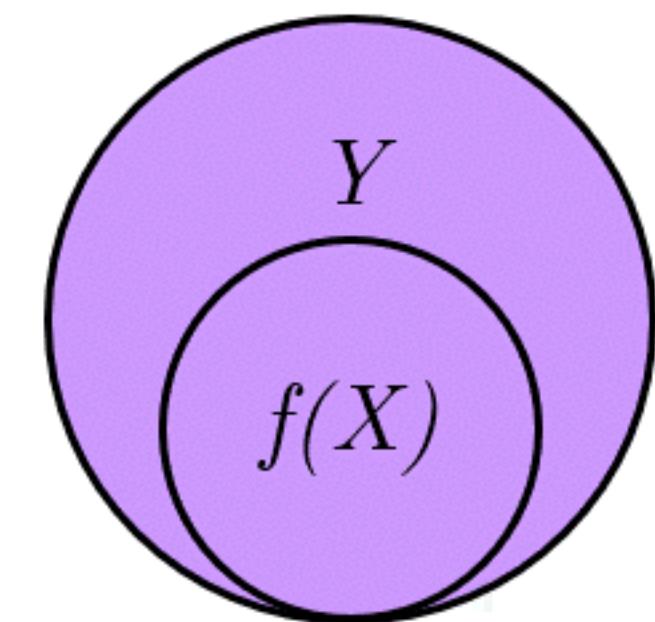
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f



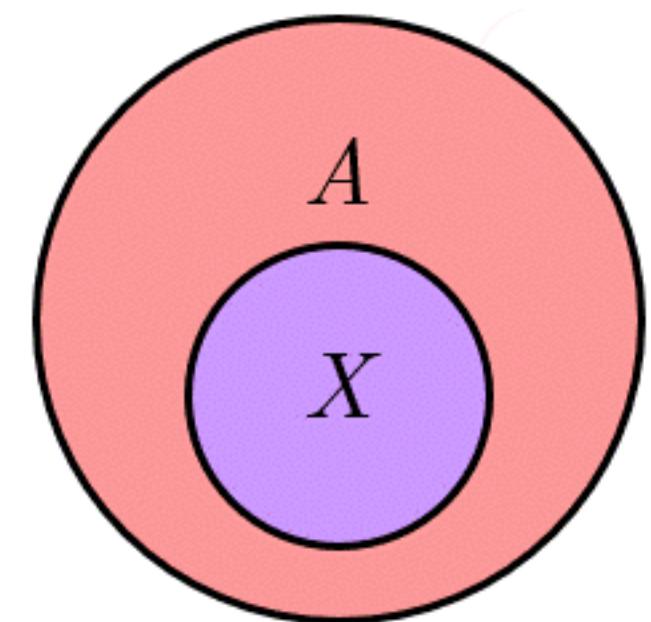
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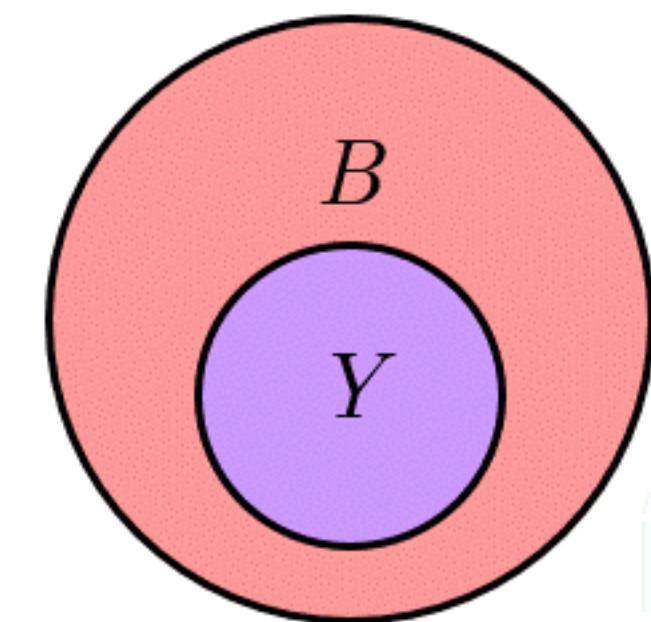
$$(X, A) \times (Y, B) := (X \times Y, A \times B)$$

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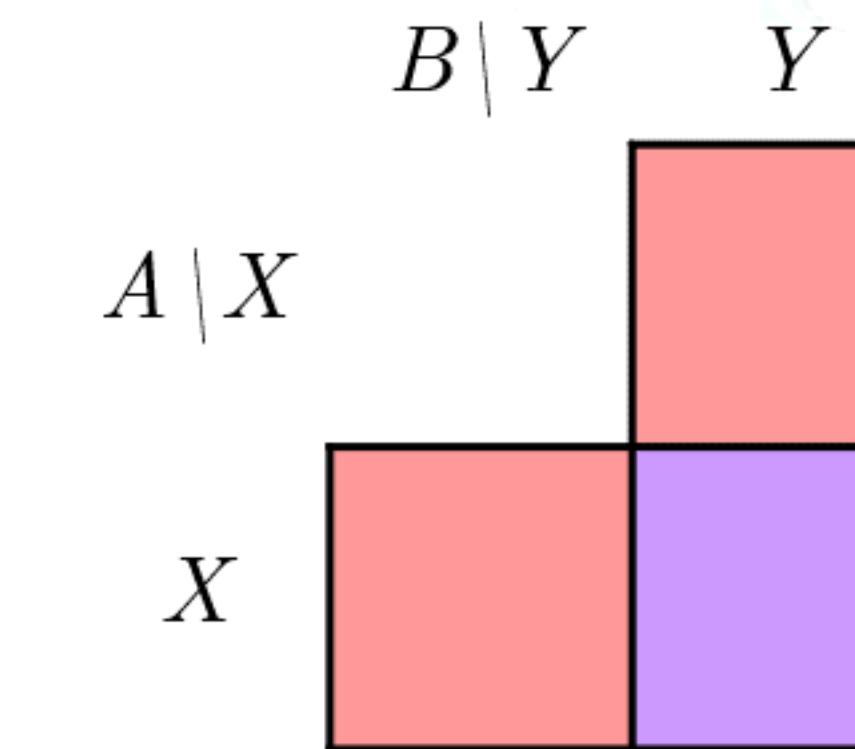
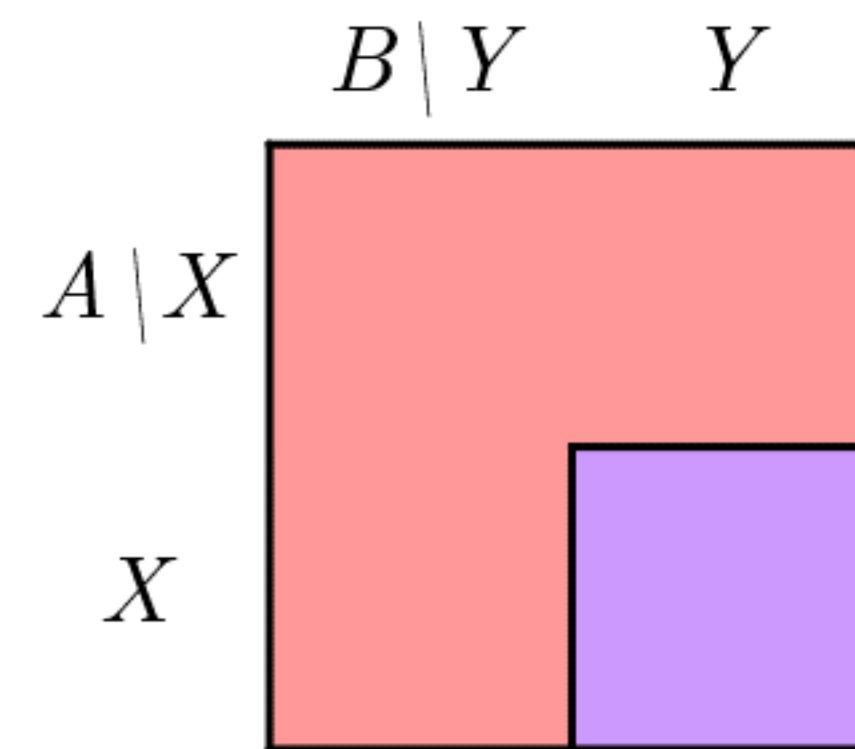
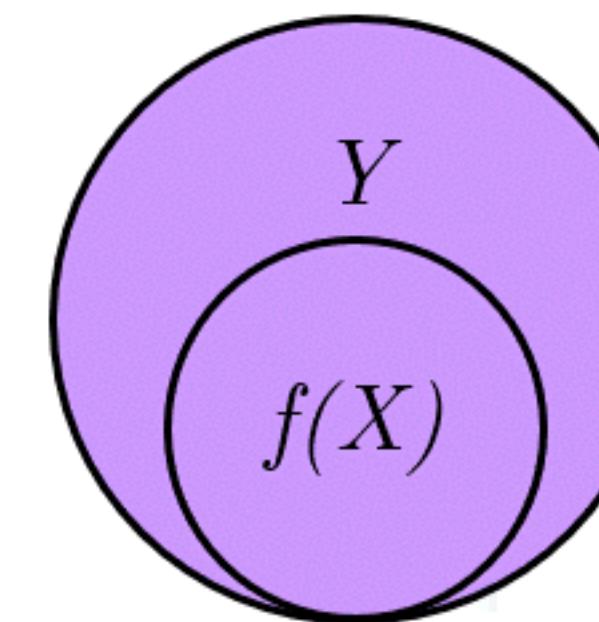
Category of Distinguished Subsets



$f \rightarrow$



such that



$$(X, A) \times (Y, B) := (X \times Y, A \times B)$$

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$(\text{Subset}, \otimes, (1, 1), \times, (1, 1))$ is duoidal

Duoidally Enriched Freyd Categories



Duoidally Enriched Freyd Categories

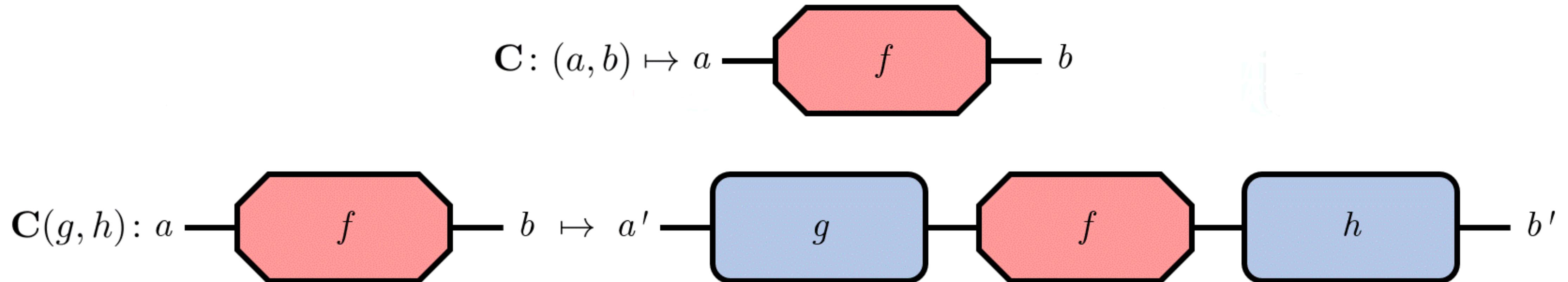
Let $(\mathbf{V}, *, J, \circ, I)$ be a duoidal category and (\mathbf{M}, \oplus, e) a monoidal category. A \mathbf{V} -Freyd category over \mathbf{M} consists of



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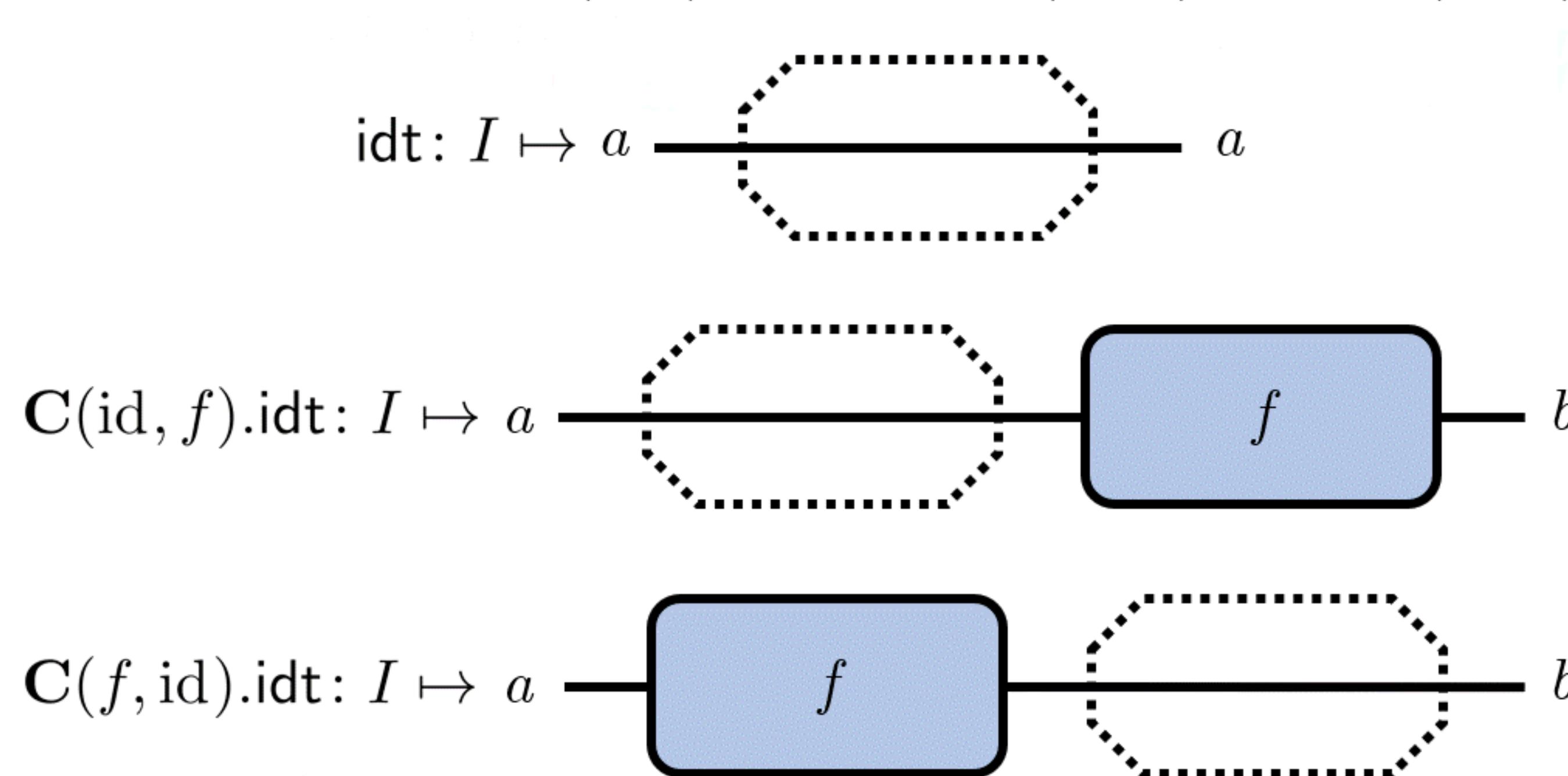
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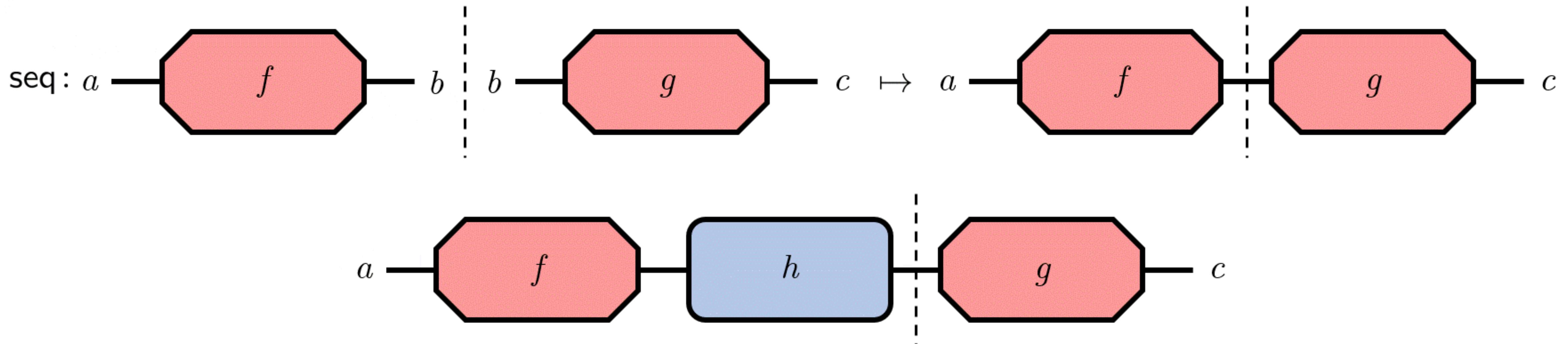
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- an extranatural family $\text{idt}: I \rightarrow \mathbf{C}(a, a)$, meaning $\mathbf{C}(\text{id}, f).\text{idt} = \mathbf{C}(f, \text{id}).\text{idt}$



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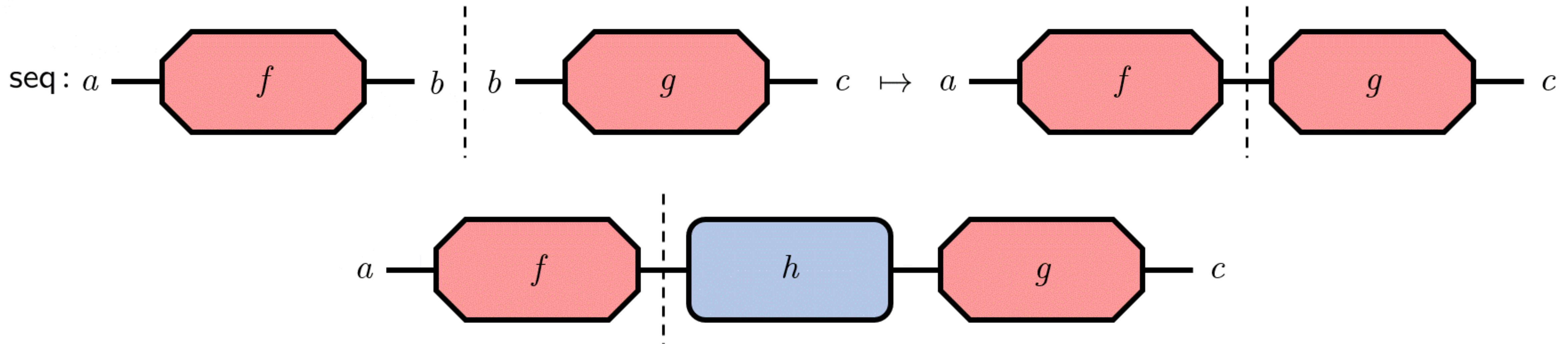
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Duoidally Enriched Freyd Categories

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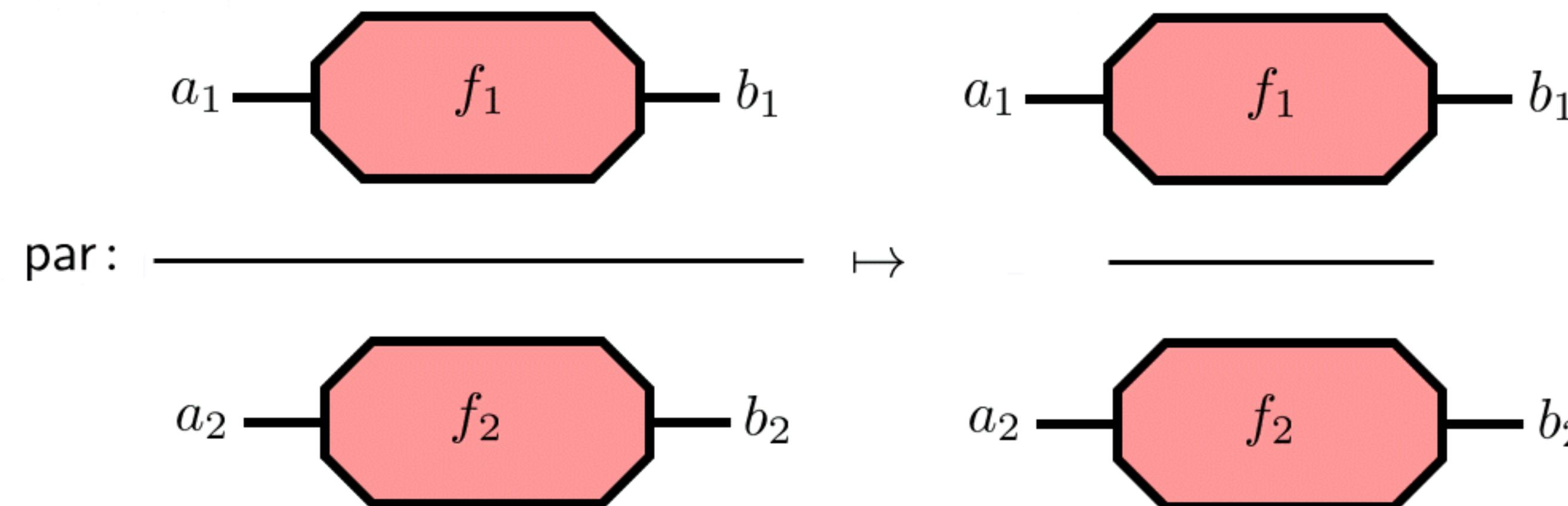
- a bifunctor $\mathbf{C}: \mathbf{M}^{\text{op}} \times \mathbf{M} \rightarrow \mathbf{V}$
- an extranatural family $\text{idt}: I \rightarrow \mathbf{C}(a, a)$, meaning $\mathbf{C}(\text{id}, f).\text{idt} = \mathbf{C}(f, \text{id}).\text{idt}$
- an extranatural family $\text{seq}: \mathbf{C}(a, b) \circ \mathbf{C}(b, c) \rightarrow \mathbf{C}(a, c)$, meaning seq is natural in a and c , and $\text{seq}.(\text{id} \circ \mathbf{C}(h, \text{id})) = \text{seq}.(\mathbf{C}(\text{id}, h) \circ \text{id})$
- a morphism $\text{zero}: J \rightarrow \mathbf{C}(e, e)$



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- a morphism $\text{zero}: J \rightarrow \mathbf{C}(e, e)$
- a natural family $\text{par}: \mathbf{C}(a_1, b_1) * \mathbf{C}(a_2, b_2) \rightarrow \mathbf{C}(a_1 \oplus a_2, b_1 \oplus b_2)$



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idt is the identity for seq , that is, $\text{seq}(\text{idt} \circ \text{id}) = \lambda$ and symmetrically

$$\begin{array}{c}
 \text{Diagram 1: } a \xrightarrow{\text{f}} b \\
 \text{Diagram 2: } a \xrightarrow{\text{f}} b \\
 \text{Diagram 3: } a \xrightarrow{\text{f}} b
 \end{array}
 = \begin{array}{c}
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The diagrams show the identity property of idt for seq . Each diagram consists of a red octagon labeled f with inputs a and b . In Diagram 1, the path from a to b is solid. In Diagram 2, the path is dashed. In Diagram 3, the path is solid. Vertical dashed lines separate the three diagrams.

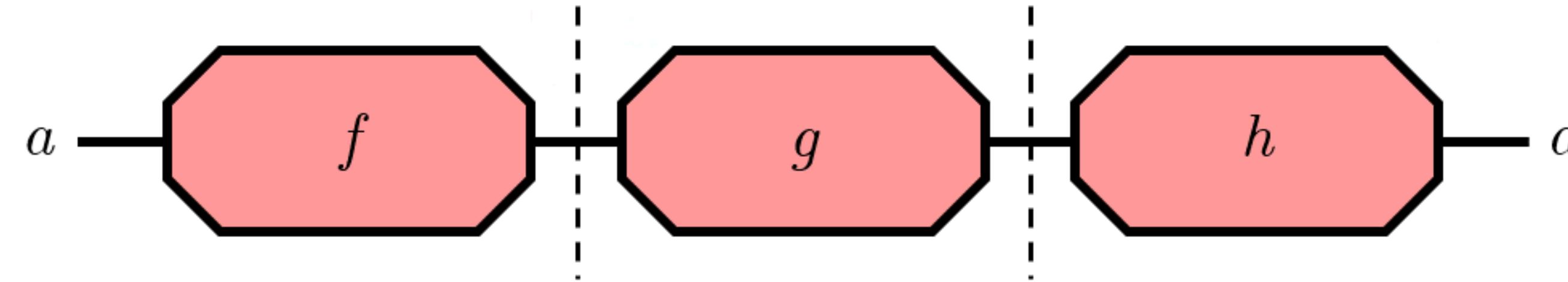
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seq is associative, that is, $\text{seq}.(\text{seq} \circ \text{id}) = \text{seq}.(\text{id} \circ \text{seq}).\alpha$

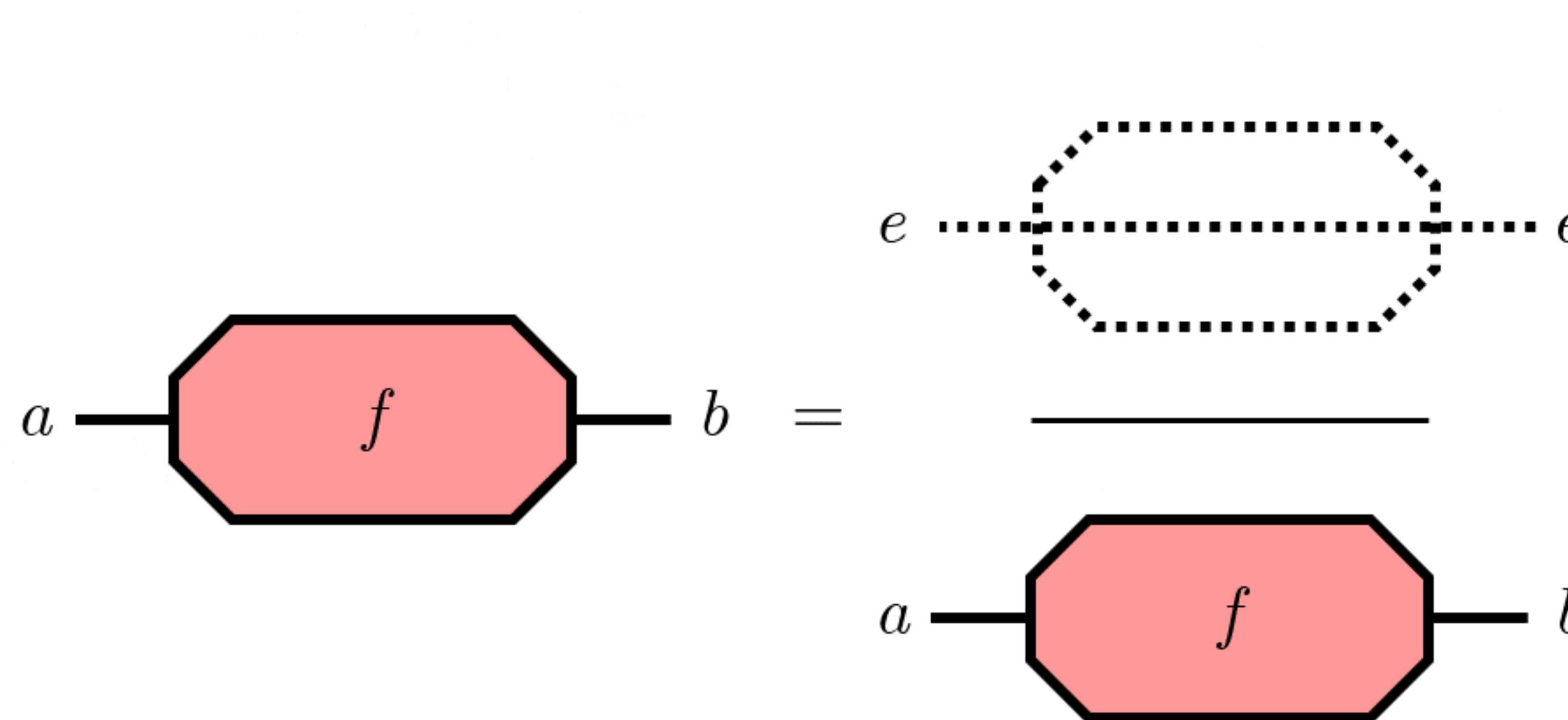


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zero is the identity for par, that is, $\mathbf{C}(\lambda^{-1}, \lambda).\text{par}.(\text{zero} * \text{id}) = \lambda$ and sym.

$$a \xrightarrow{f} b = \frac{a \xrightarrow{\quad} b}{\text{zero}}$$


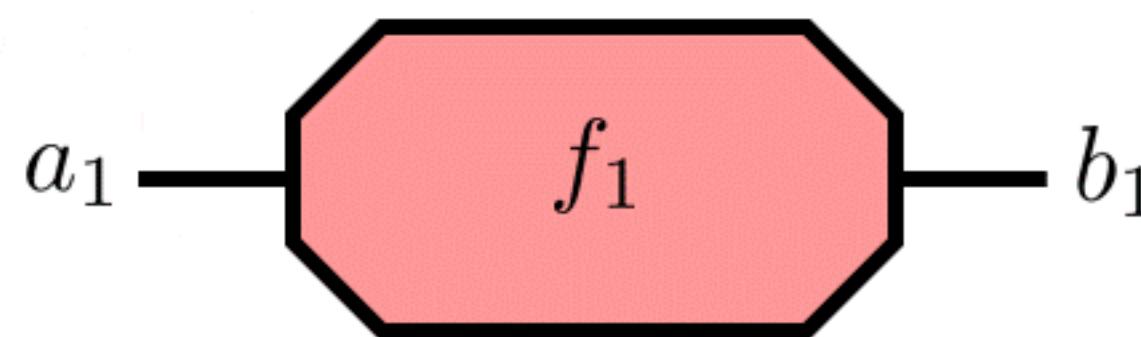
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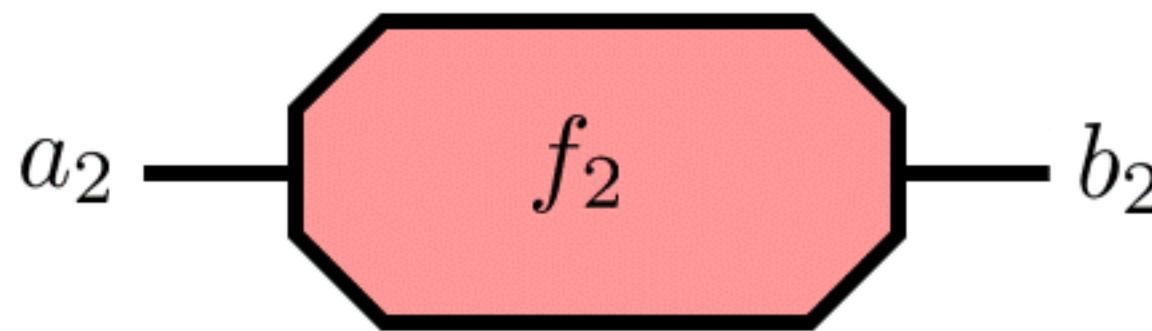
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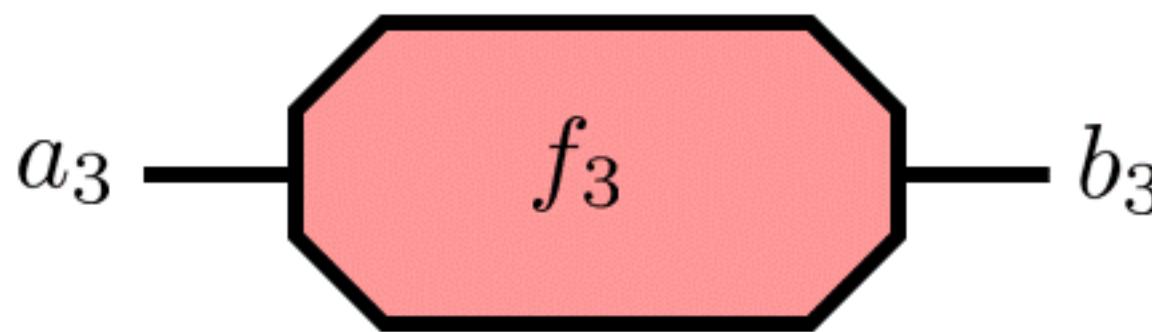
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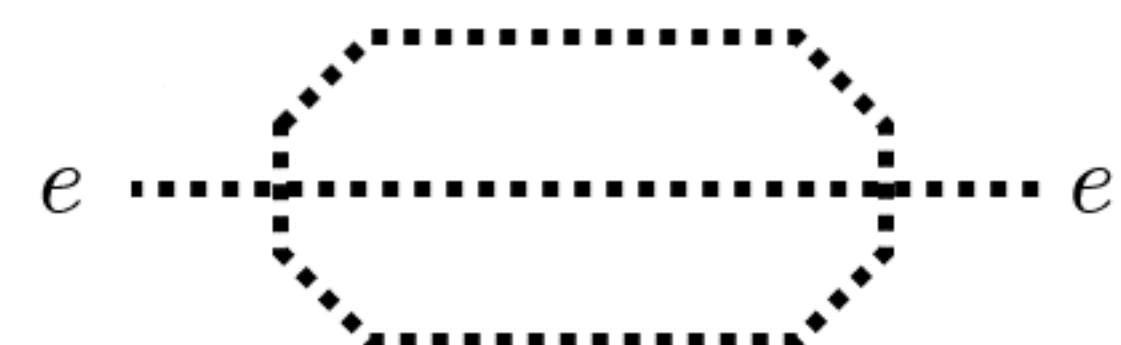


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`idt` respects `zero` via `idt.ε = zero`



$$*\frac{A \circ B}{C \circ D} \xrightarrow{\zeta} \begin{array}{c} \circ \\ A \\ * \\ C \end{array} \begin{array}{c} B \\ * \\ D \end{array} \quad J \xrightarrow{\Delta} \begin{array}{c} \circ \\ J \\ | \\ J \end{array} \quad * \frac{I}{I} \xrightarrow{\nabla} I \quad J \xrightarrow{\epsilon} I$$

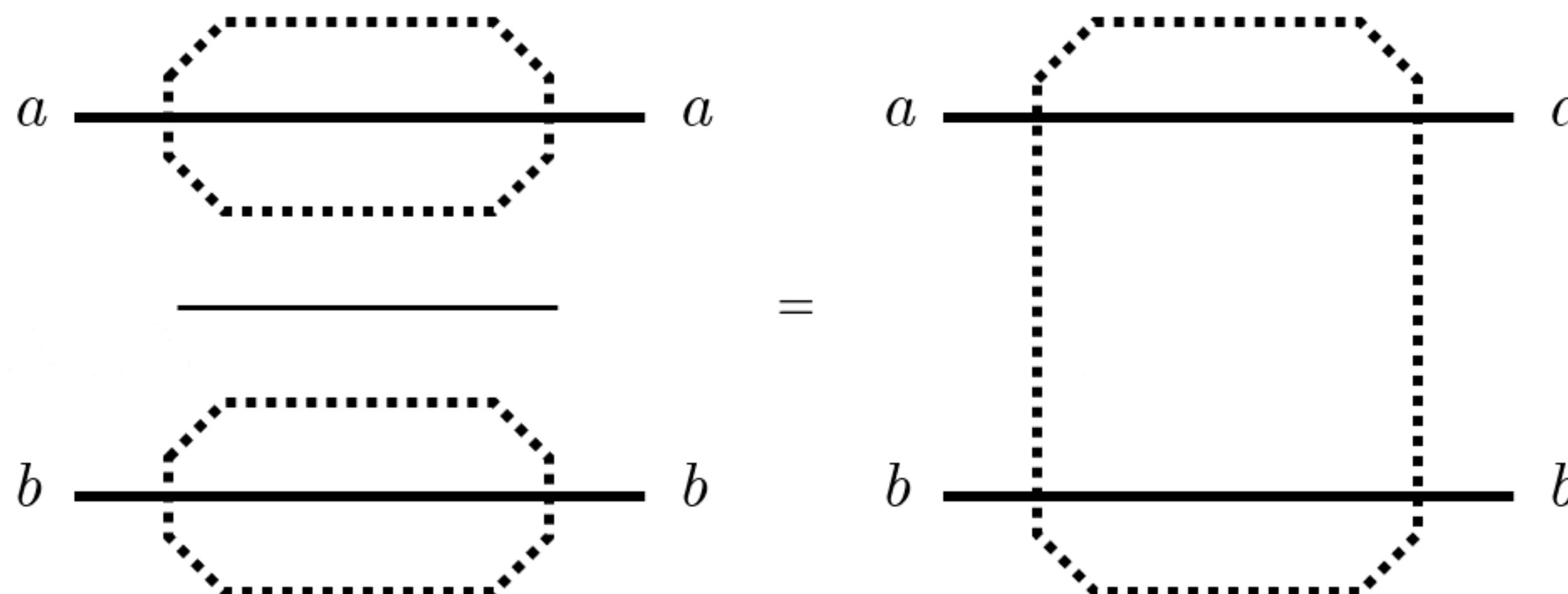
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idt respects par via $\text{idt}. \nabla = \text{par}. (\text{idt} * \text{idt})$



$$*\frac{A \circ B}{C \circ D} \xrightarrow{\zeta} \begin{array}{c} \circ \\ A \end{array} \begin{array}{c} B \\ * \\ C \end{array} \begin{array}{c} D \\ * \end{array} \quad J \xrightarrow{\Delta} \begin{array}{c} \circ \\ J \end{array} \begin{array}{c} J \\ | \\ J \end{array} \quad *\frac{I}{I} \xrightarrow{\nabla} I \quad J \xrightarrow{\epsilon} I$$

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seq respects zero via $\text{seq}.\text{(zero} \circ \text{zero}).\Delta = \text{zero}$

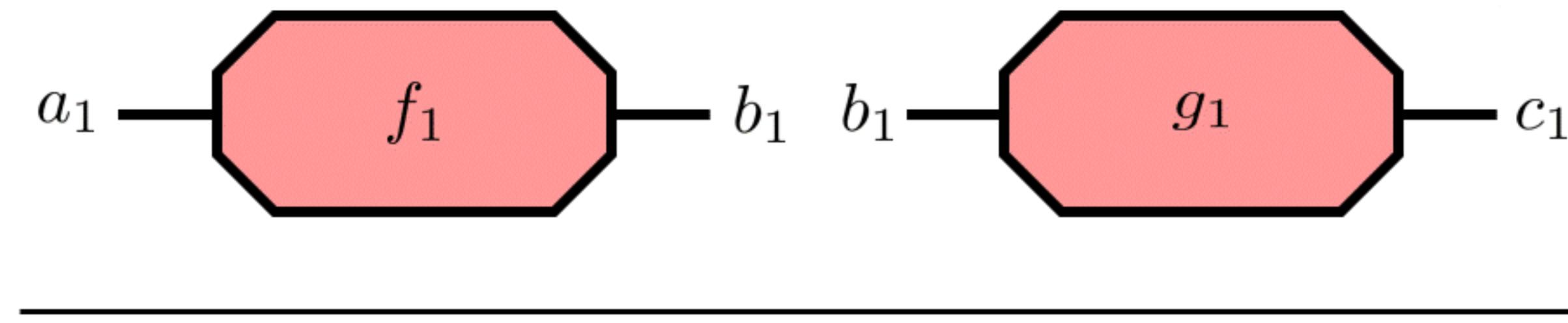
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seq respects par via $\text{seq}.(\text{par} \circ \text{par}).\zeta = \text{par}.(\text{seq} * \text{seq})$



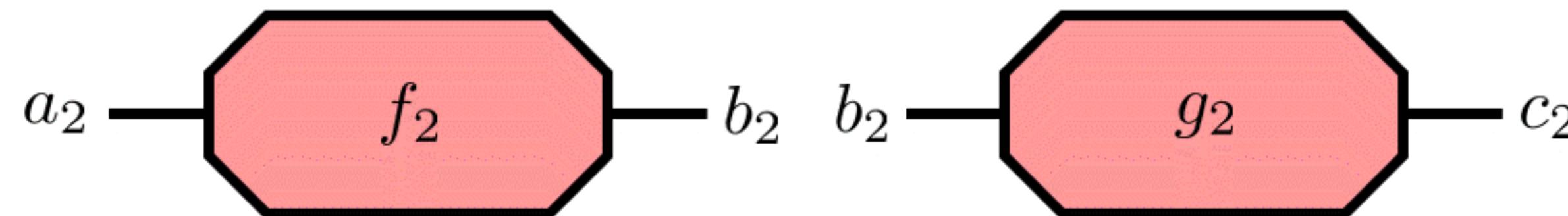
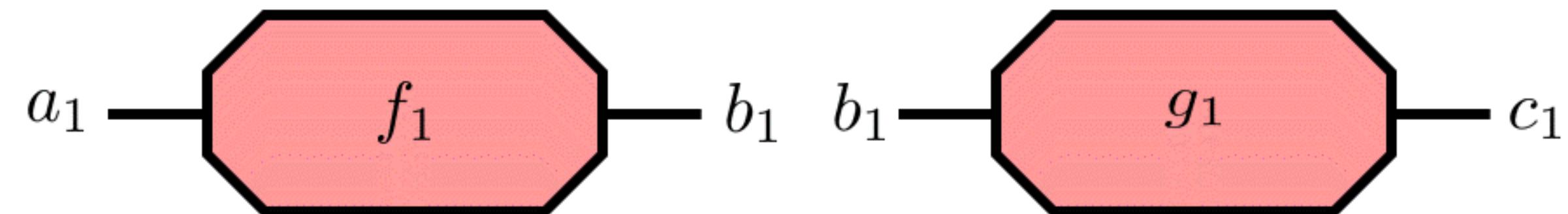
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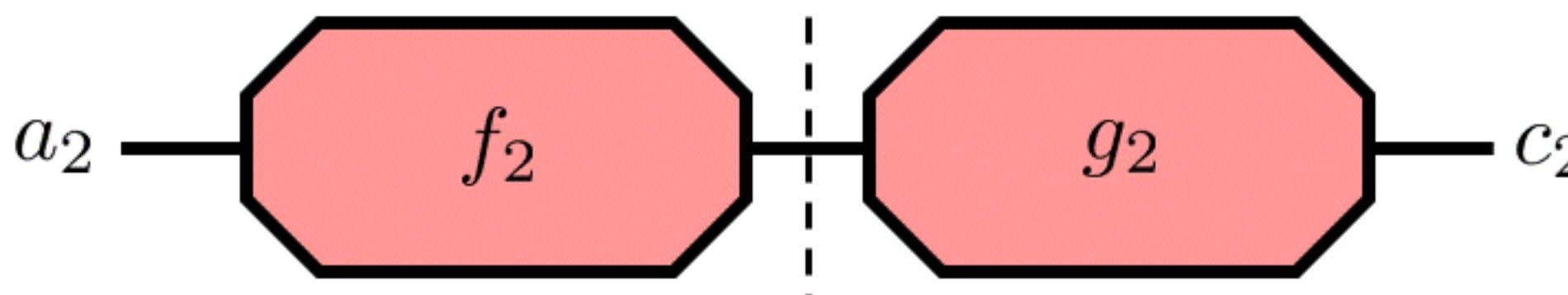
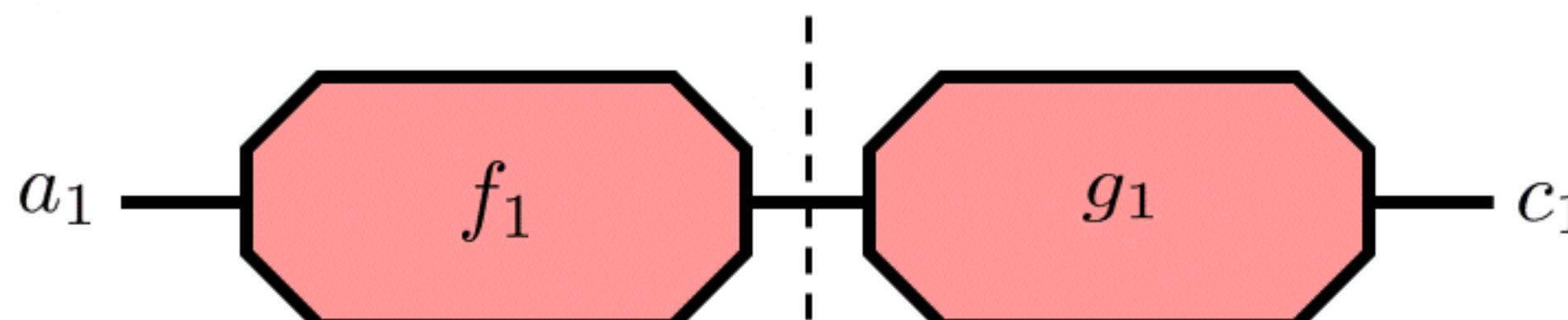
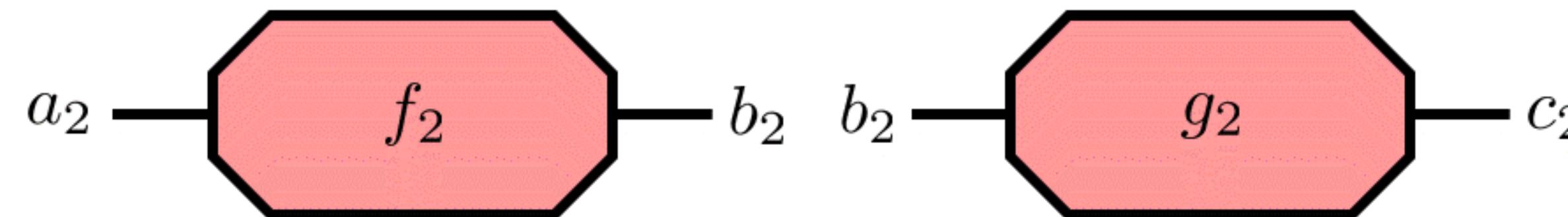
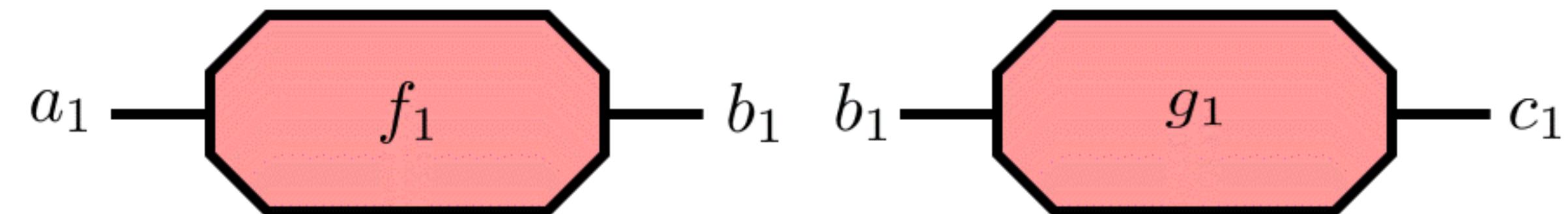


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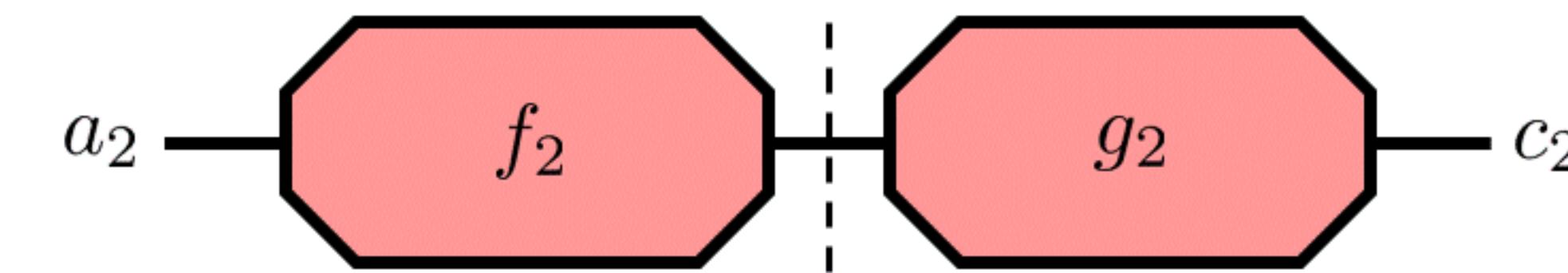
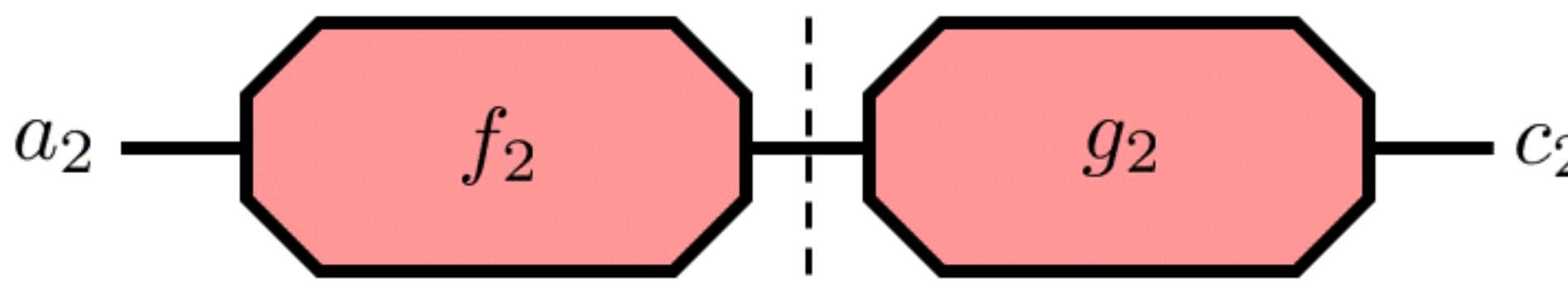
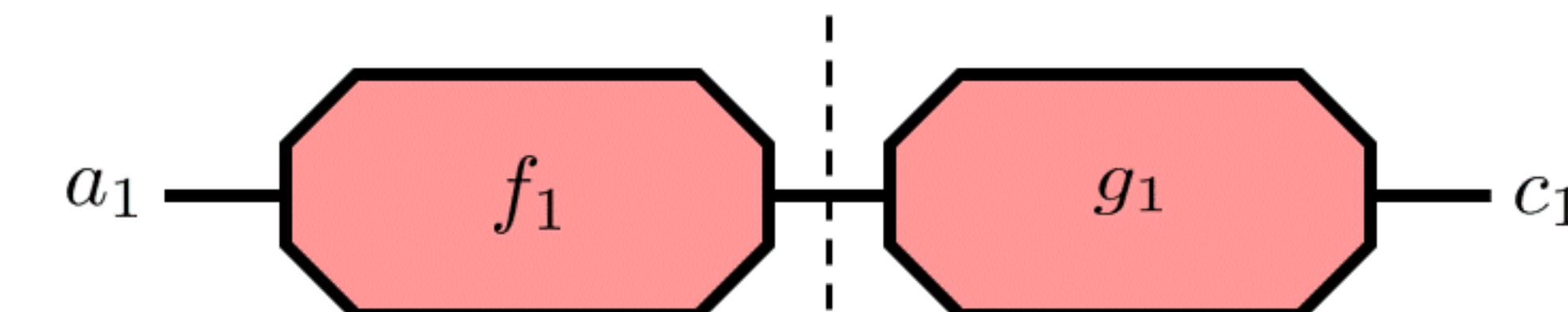
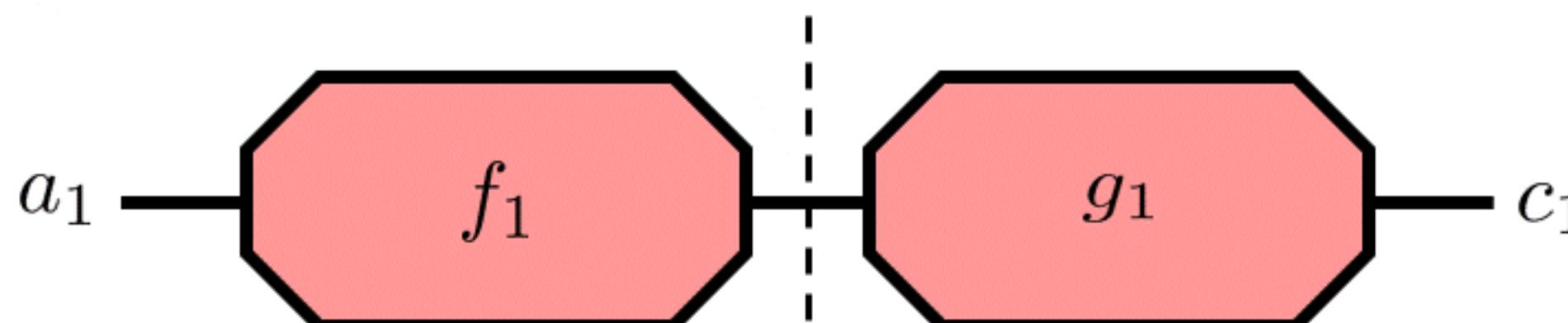
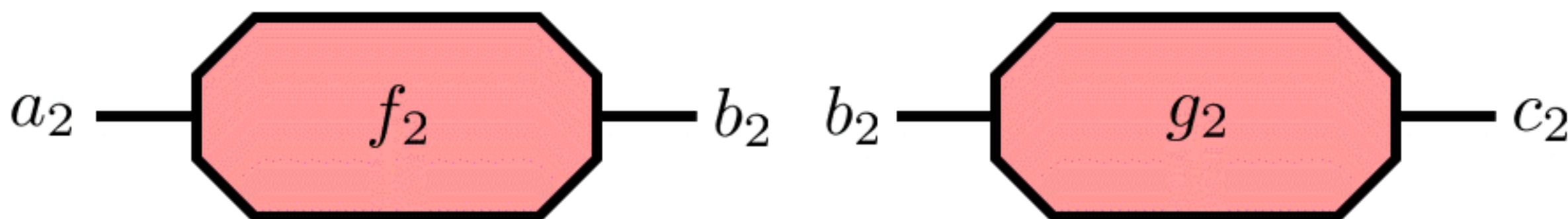
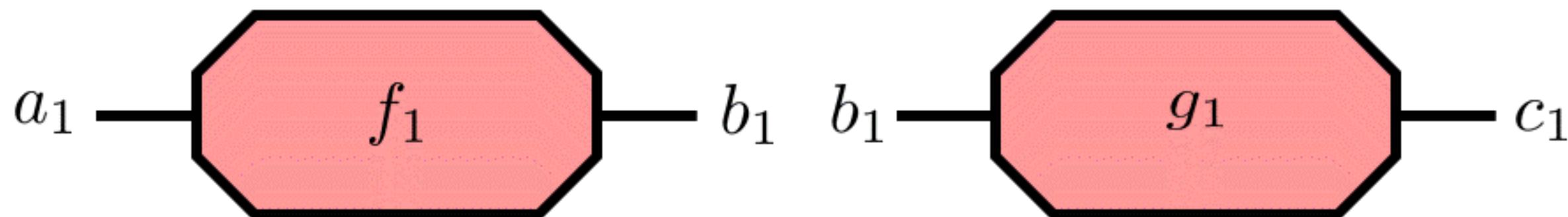
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Compound State

Compound state is a state where the system has multiple components. It is a generalization of the concept of a state in a system. In a compound state, the system is composed of multiple components, each of which has its own state. The overall state of the system is determined by the states of all of its components.



Compound State

$$R \coloneqq \{\mathbb{B}, \mathbb{Z}, \dots\}$$

$$\mathcal{P}_f(R) = \{\emptyset, \{\mathbb{B}\}, \{\mathbb{Z}\}, \{\mathbb{B}, \mathbb{Z}\}, \dots\}$$

$$\mathbf{V} = (\mathbf{Label}, \parallel, \text{cst}_{\emptyset}, \cup, \text{cst}_{\emptyset})$$

$$\mathbf{M} = (\mathbf{Set}, \times, 1)$$

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$$\mathbf{C}(a, b) \coloneqq \left(\sum_{Q \in \mathcal{P}_f(R)} \mathbf{Set}(\Pi_Q \times a, \Pi_Q \times b) \right) \xrightarrow{\ell} \mathcal{P}_f(R)$$
$$(Q, f) \mapsto Q$$

$$\Pi_Q \coloneqq \Pi_{x \in Q} x \text{ for } Q \in \mathcal{P}_f(R), \text{ e.g. } \Pi_{\{\mathbb{B}, \mathbb{Z}\}} = \mathbb{B} \times \mathbb{Z}$$

Compound State

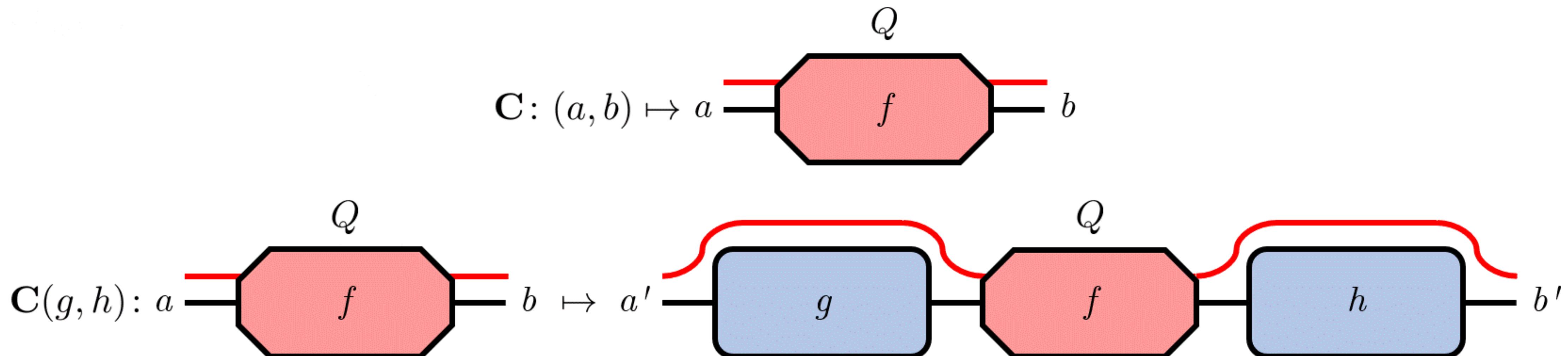
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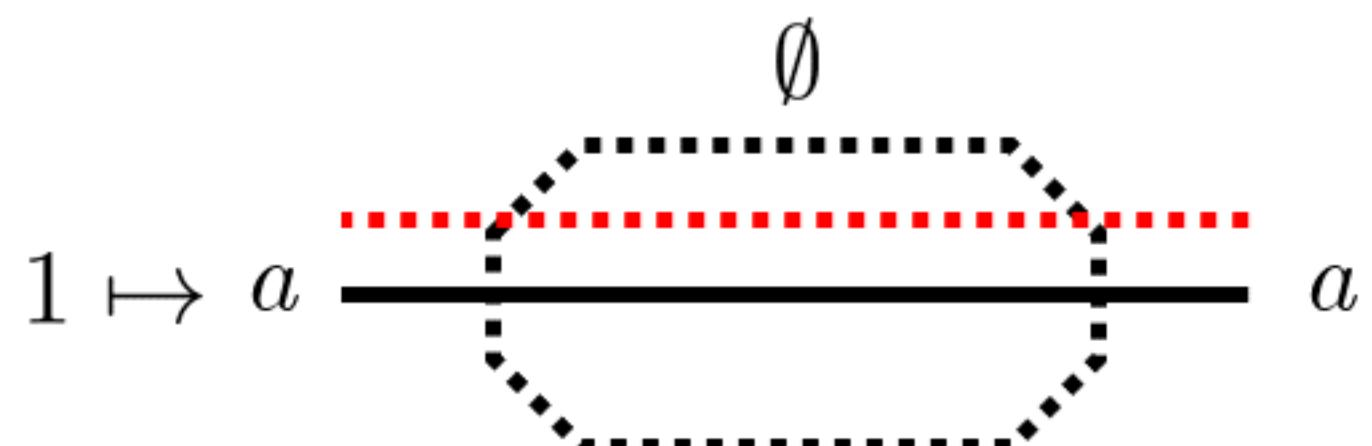
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$$\text{idt}: \text{cst}_{\emptyset} \rightarrow \mathbf{C}(a, a)$$



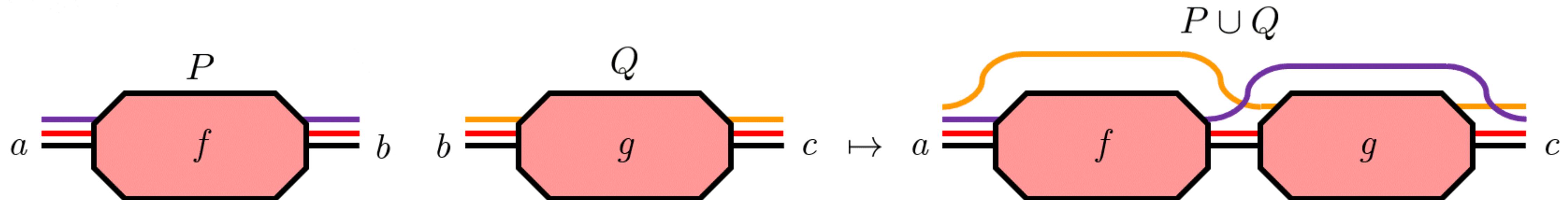
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$$\text{seq}: \mathbf{C}(a, b) \cup \mathbf{C}(b, c) \rightarrow \mathbf{C}(a, c)$$



Compound State

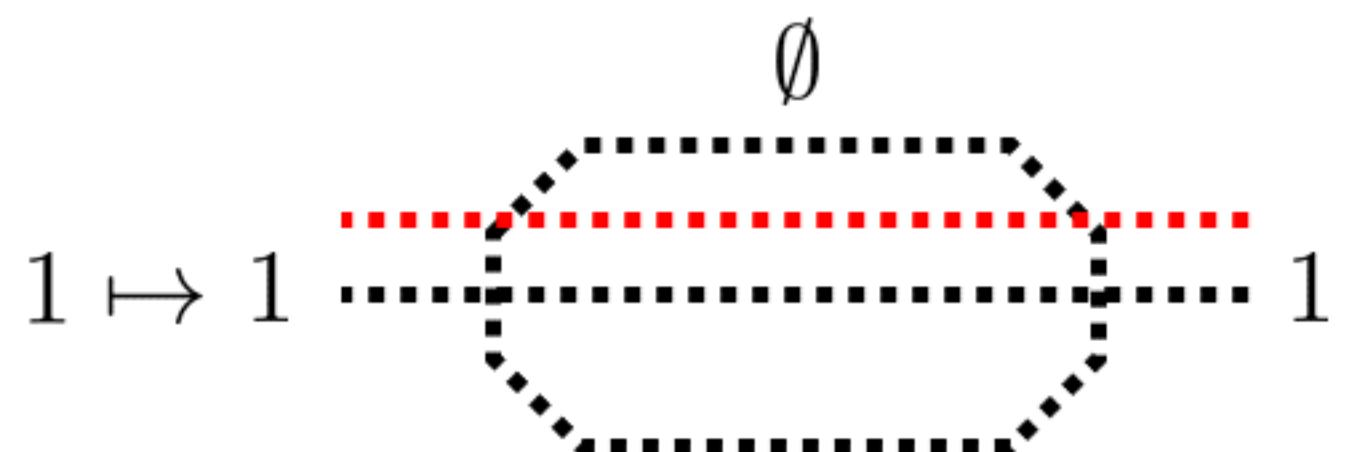
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$$\text{zero}: \text{cst}_{\emptyset} \rightarrow \mathbf{C}(1, 1)$$



Compound State

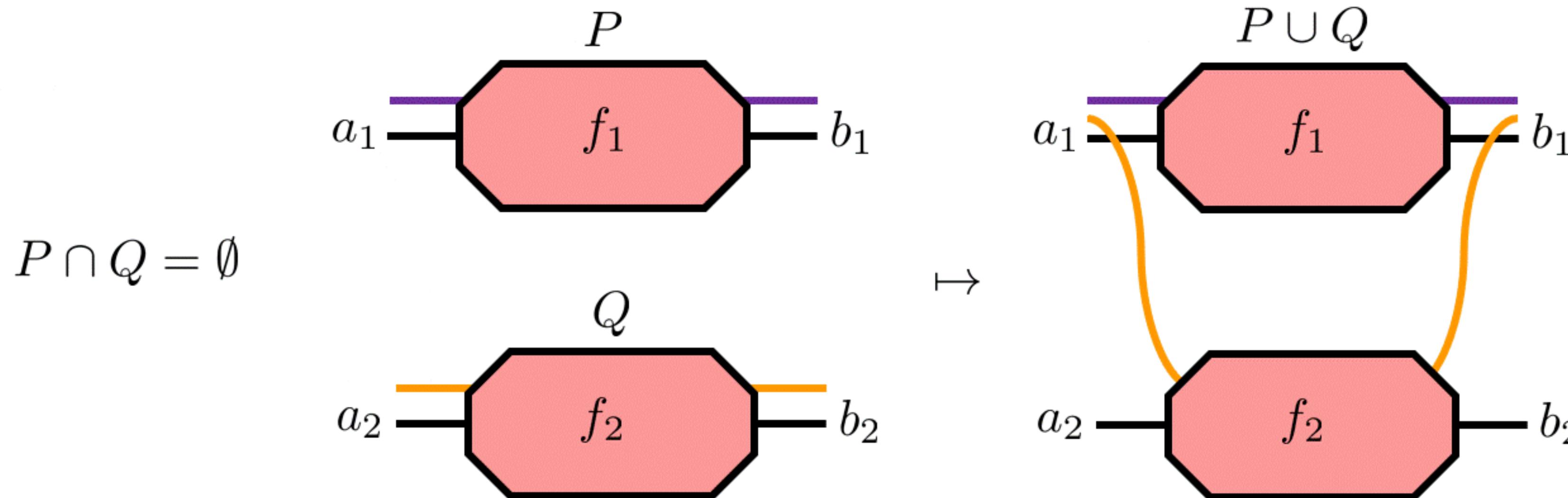
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$$\text{par}: \mathbf{C}(a_1, b_1) \parallel \mathbf{C}(a_2, b_2) \rightarrow \mathbf{C}(a_1 \times a_2, b_1 \times b_2)$$

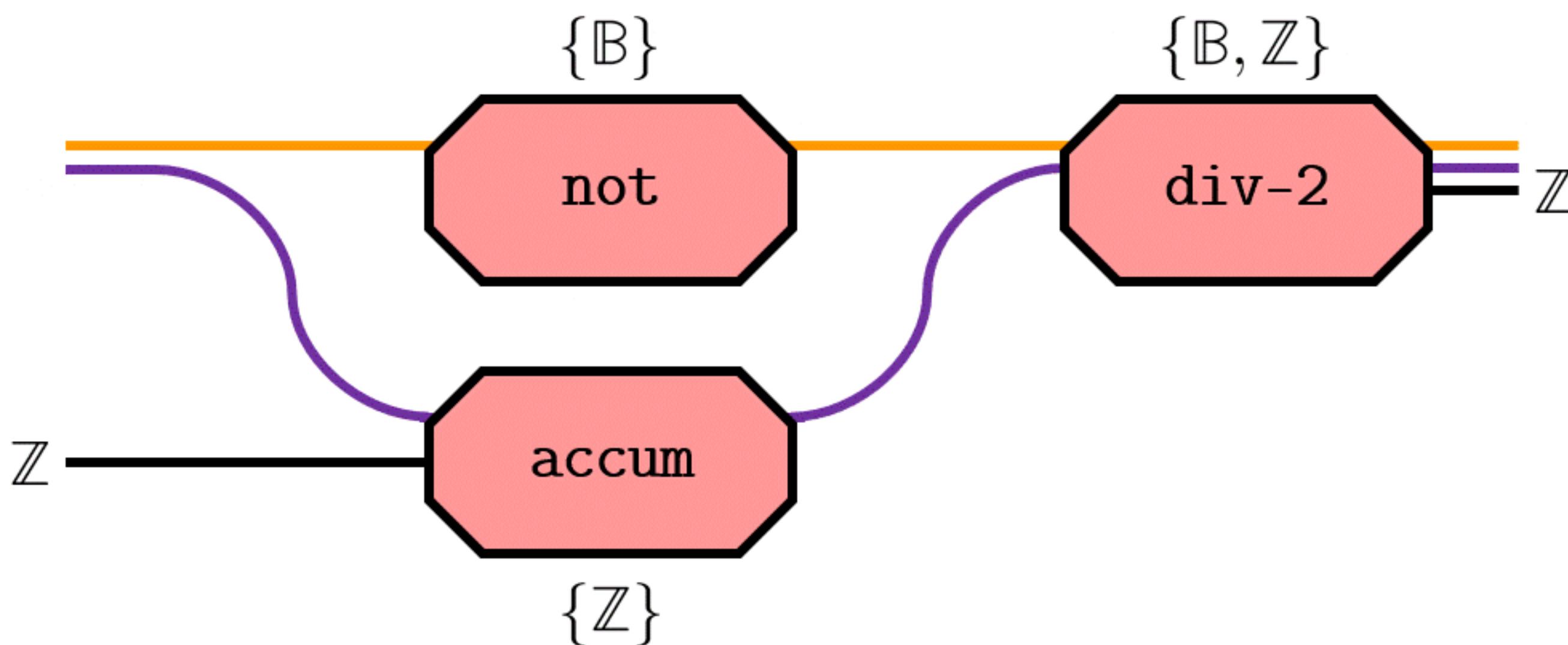


Compound State

$$R \coloneqq \{\mathbb{B}, \mathbb{Z}, \dots\} \quad \mathcal{P}_f(R) = \{\emptyset, \{\mathbb{B}\}, \{\mathbb{Z}\}, \{\mathbb{B}, \mathbb{Z}\}, \dots\}$$

$$\mathbf{V} = (\mathbf{Label}, \parallel, \text{cst}_{\emptyset}, \cup, \text{cst}_{\emptyset}) \quad \mathbf{M} = (\mathbf{Set}, \times, 1)$$

$$\mathbf{C}(a, b) := \left(\sum_{Q \in \mathcal{P}_f(R)} \mathbf{Set}(\Pi_Q \times a, \Pi_Q \times b) \right) \xrightarrow{\ell} \mathcal{P}_f(R)$$



not : () \rightarrow ()
 $\text{not}(b, ()) = (\neg b, ())$

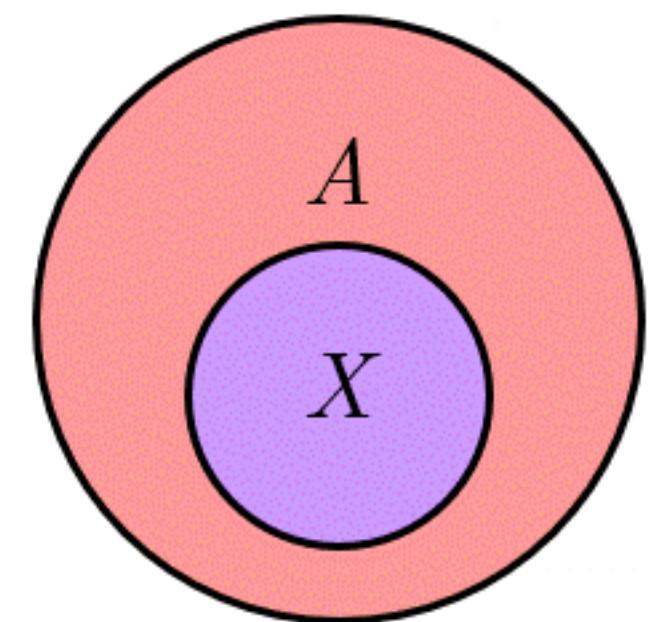
accum : $\mathbb{Z} \rightarrow ()$
 $\text{accum}(s, y) = (y + s, ())$

div-2 : () $\rightarrow \mathbb{Z}$
 $\text{div-2}(b, s, ()) =$
if b **then**
 $\quad (b, s, \text{ceil}(s/2))$
else
 $\quad (b, s, \text{floor}(s/2))$

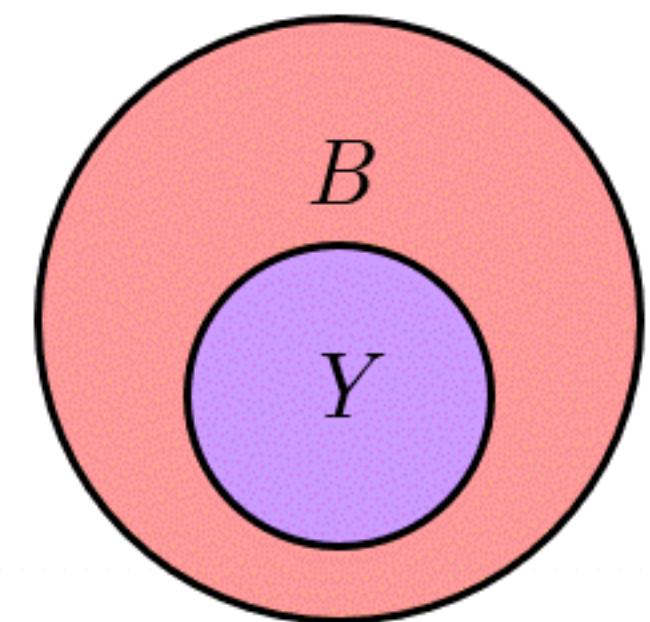
Freyd and Subset-Freyd

John Freyd
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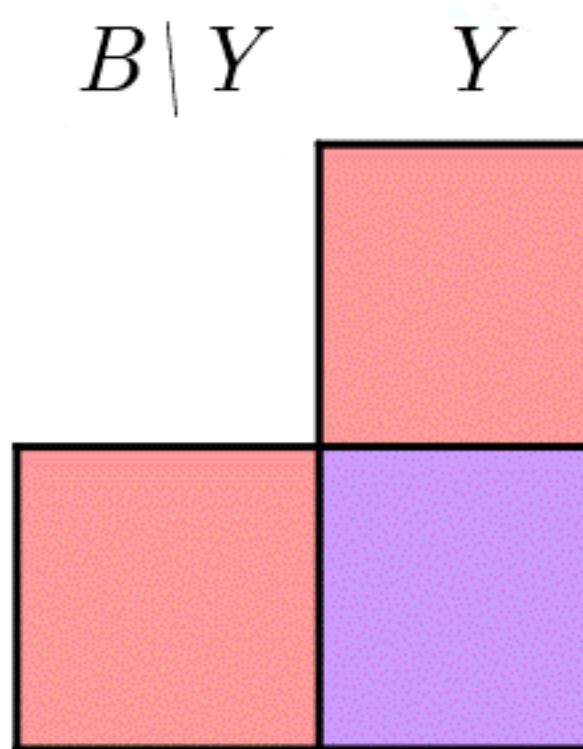
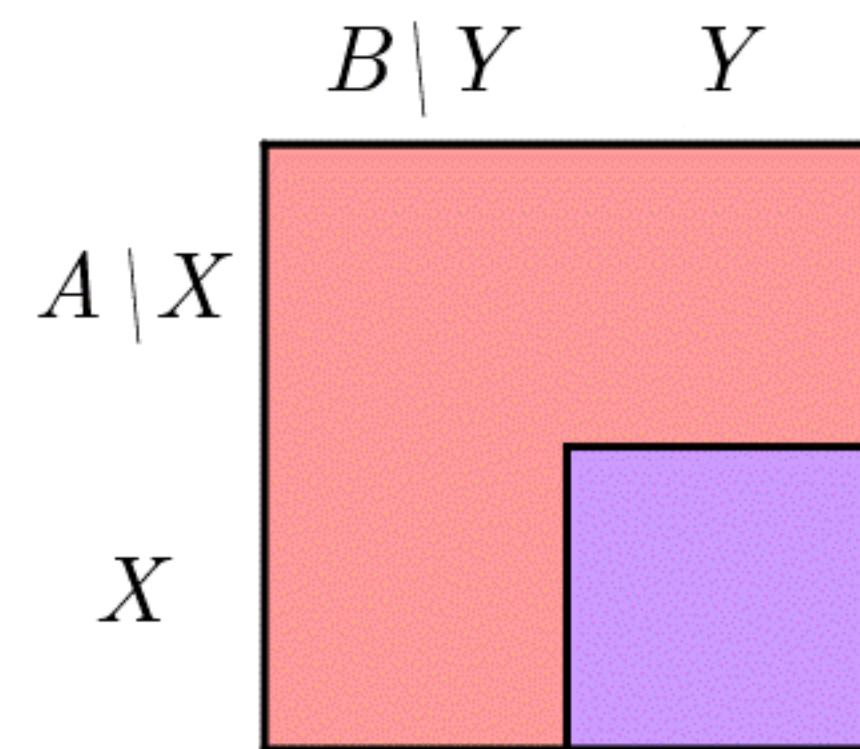
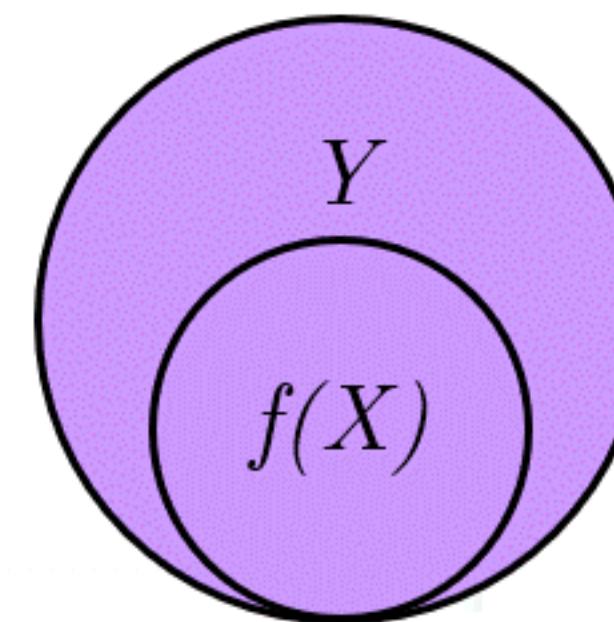
Freyd and Subset-Freyd



$f \rightarrow$



such that

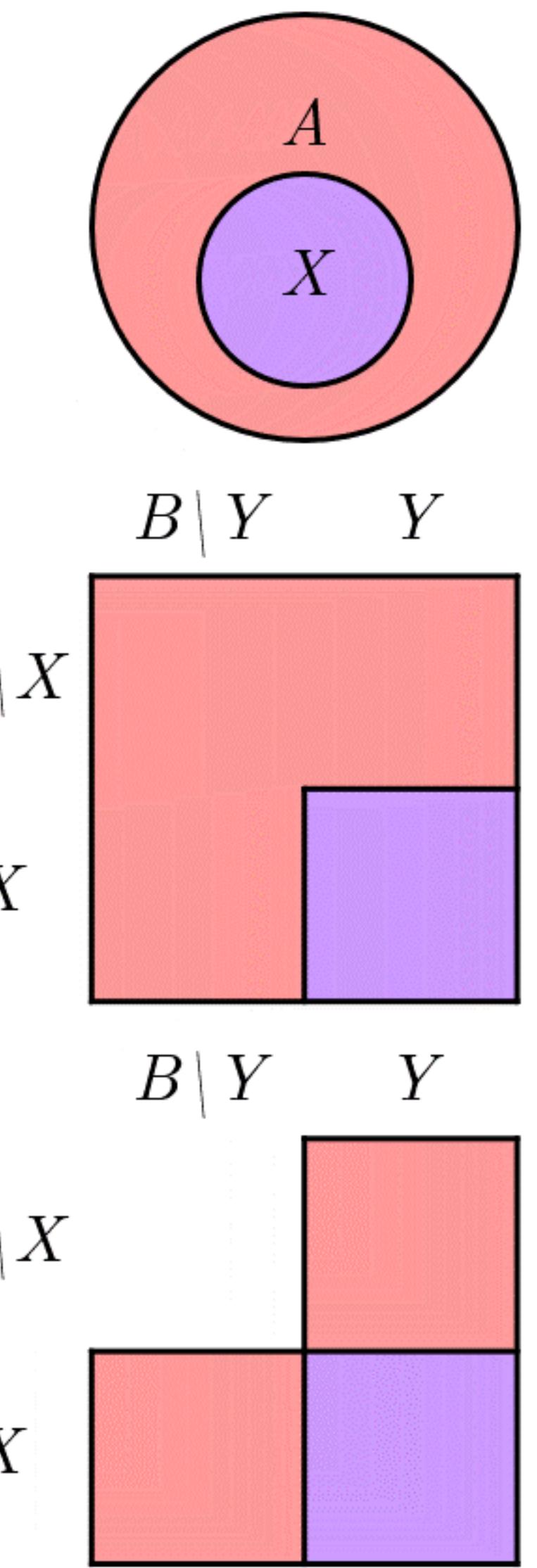
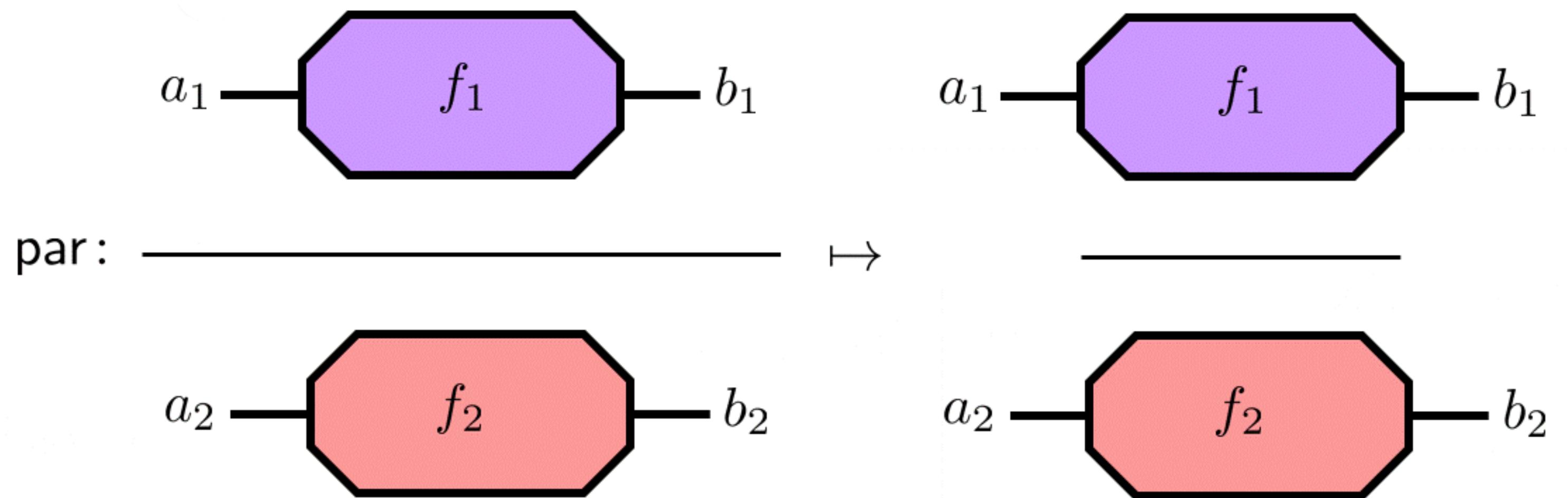


$$(X, A) \times (Y, B) := (X \times Y, A \times B)$$

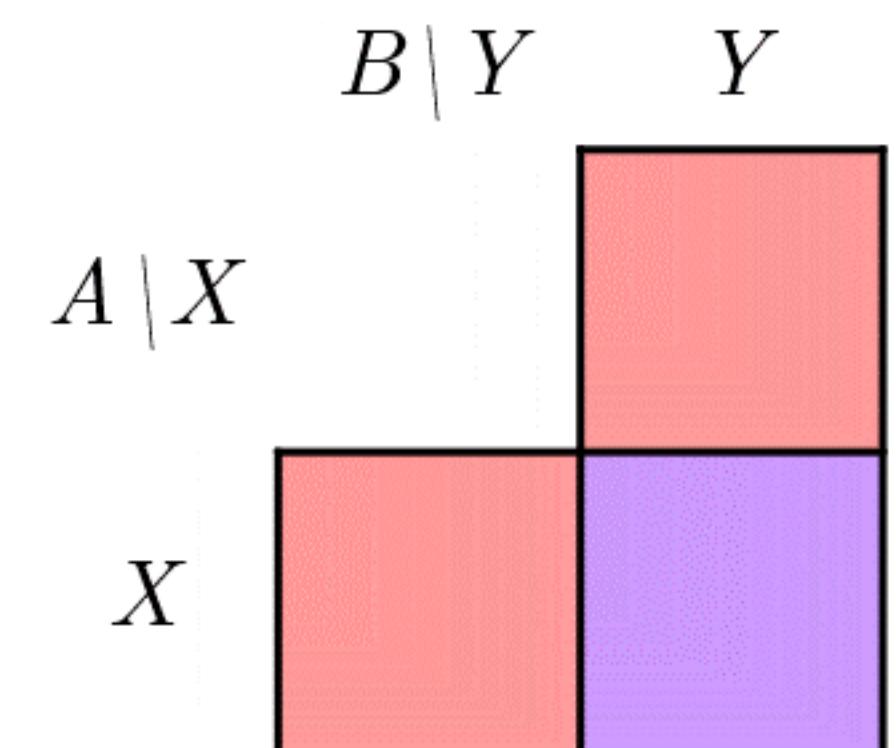
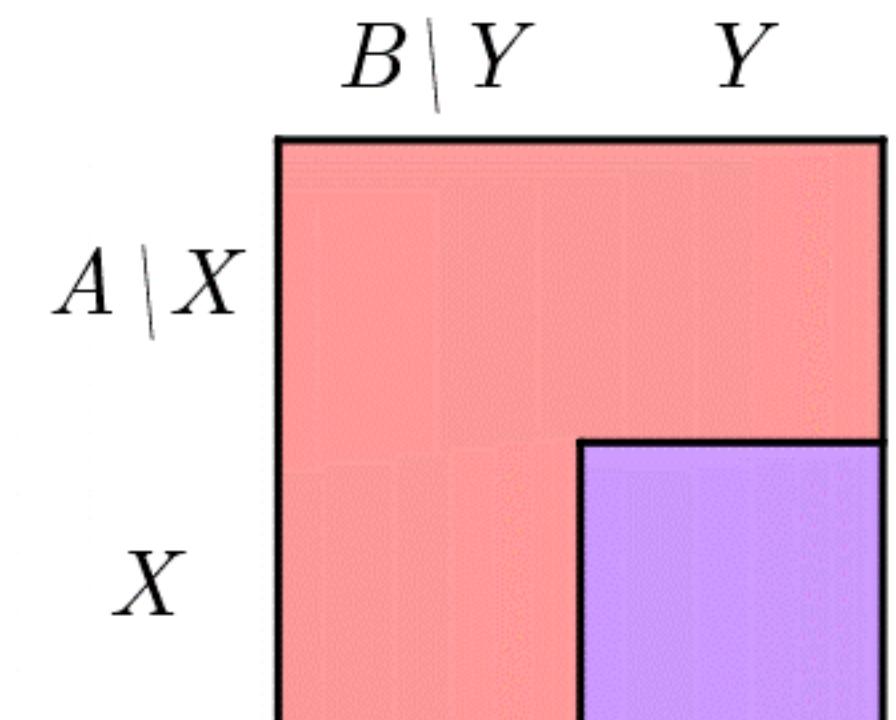
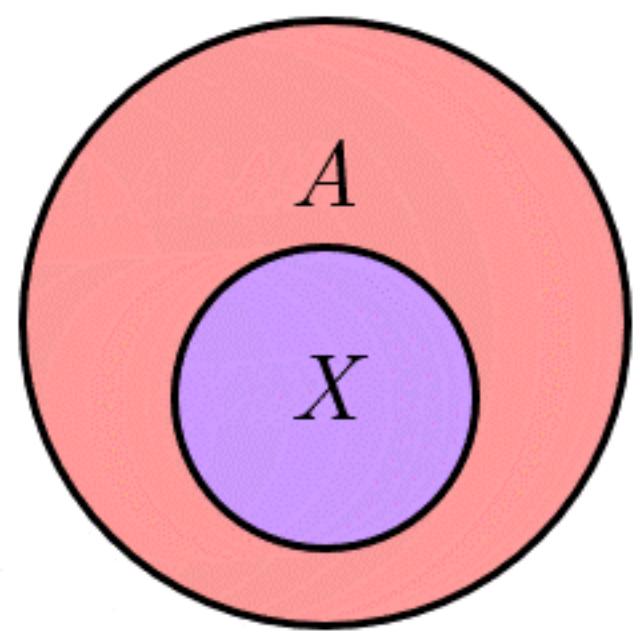
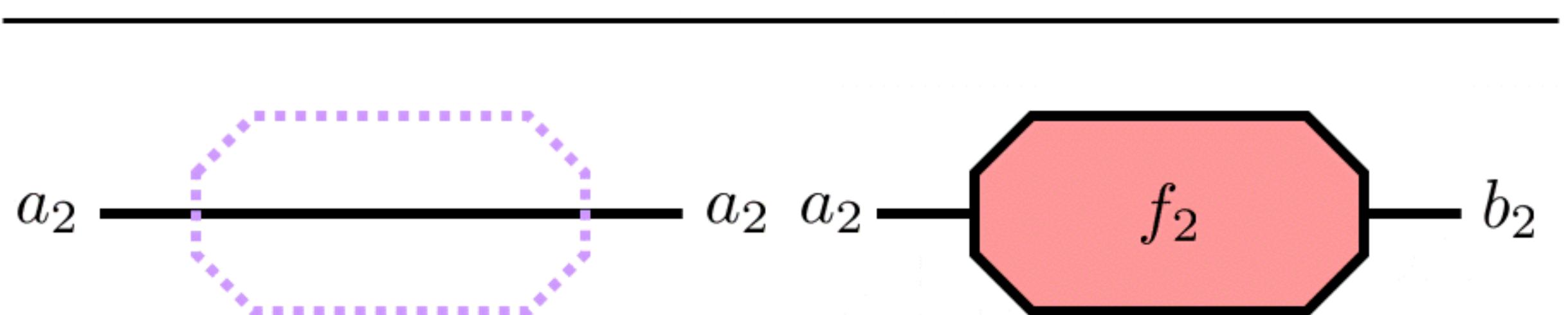
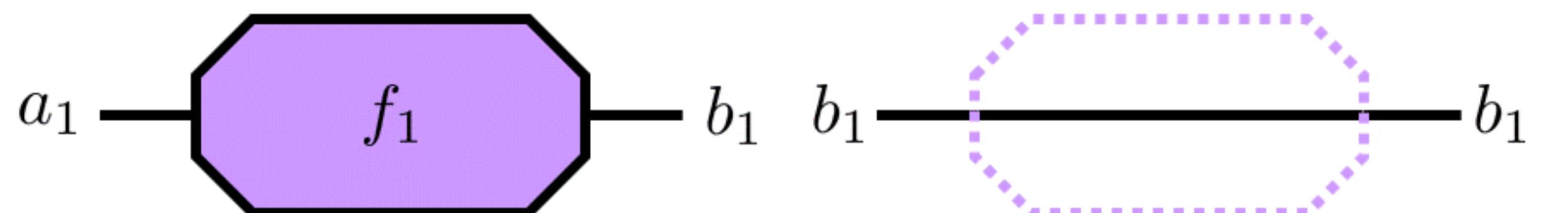
$$(X, A) \otimes (Y, B) := (X \times Y, (A \times Y) \cup (X \times B))$$

$(\text{Subset}, \otimes, (1, 1), \times, (1, 1))$ is duoidal

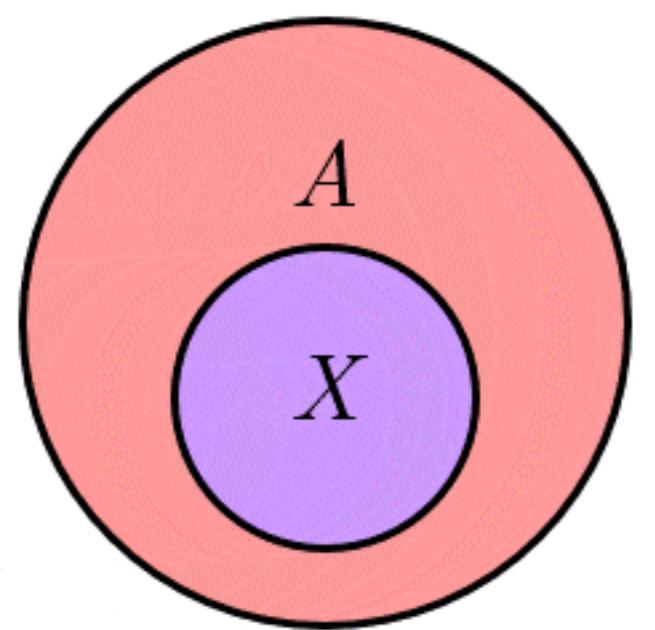
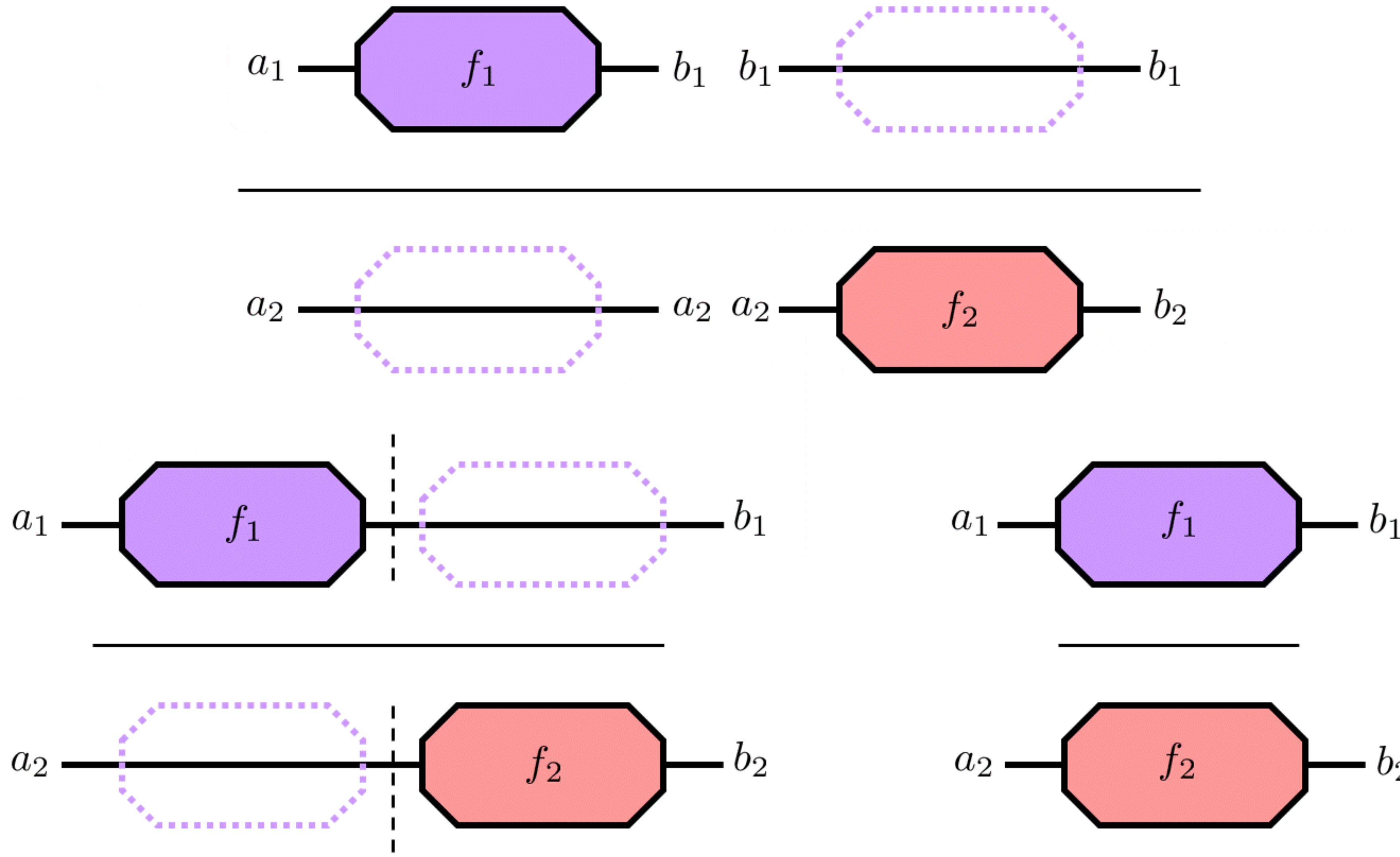
Freyd and Subset-Freyd



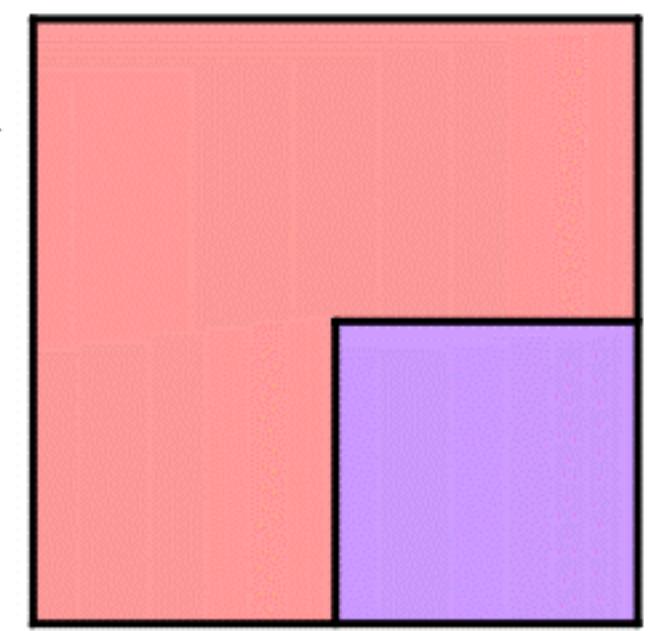
Freyd and Subset-Freyd



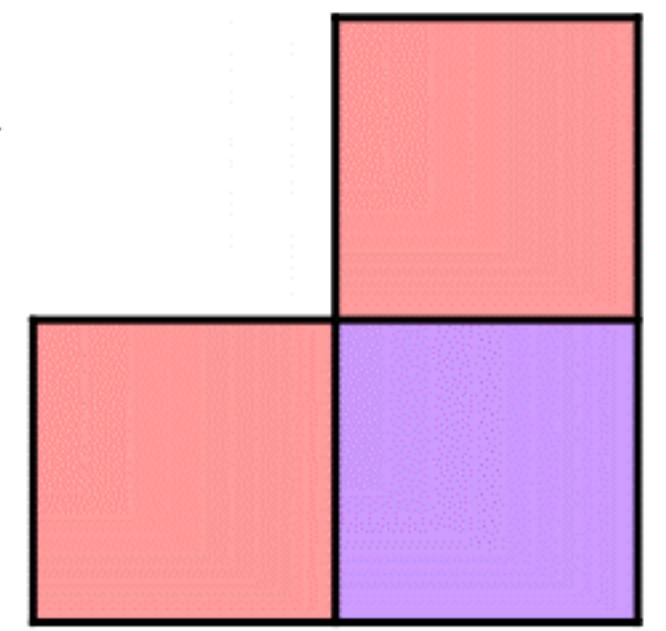
Freyd and Subset-Freyd



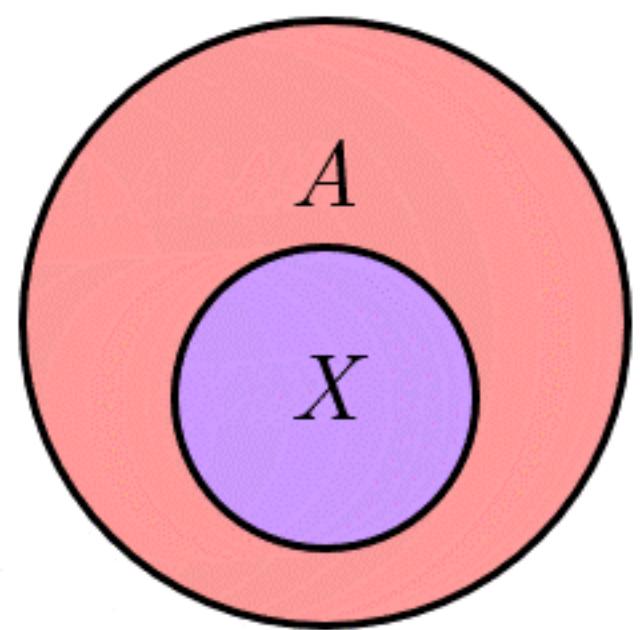
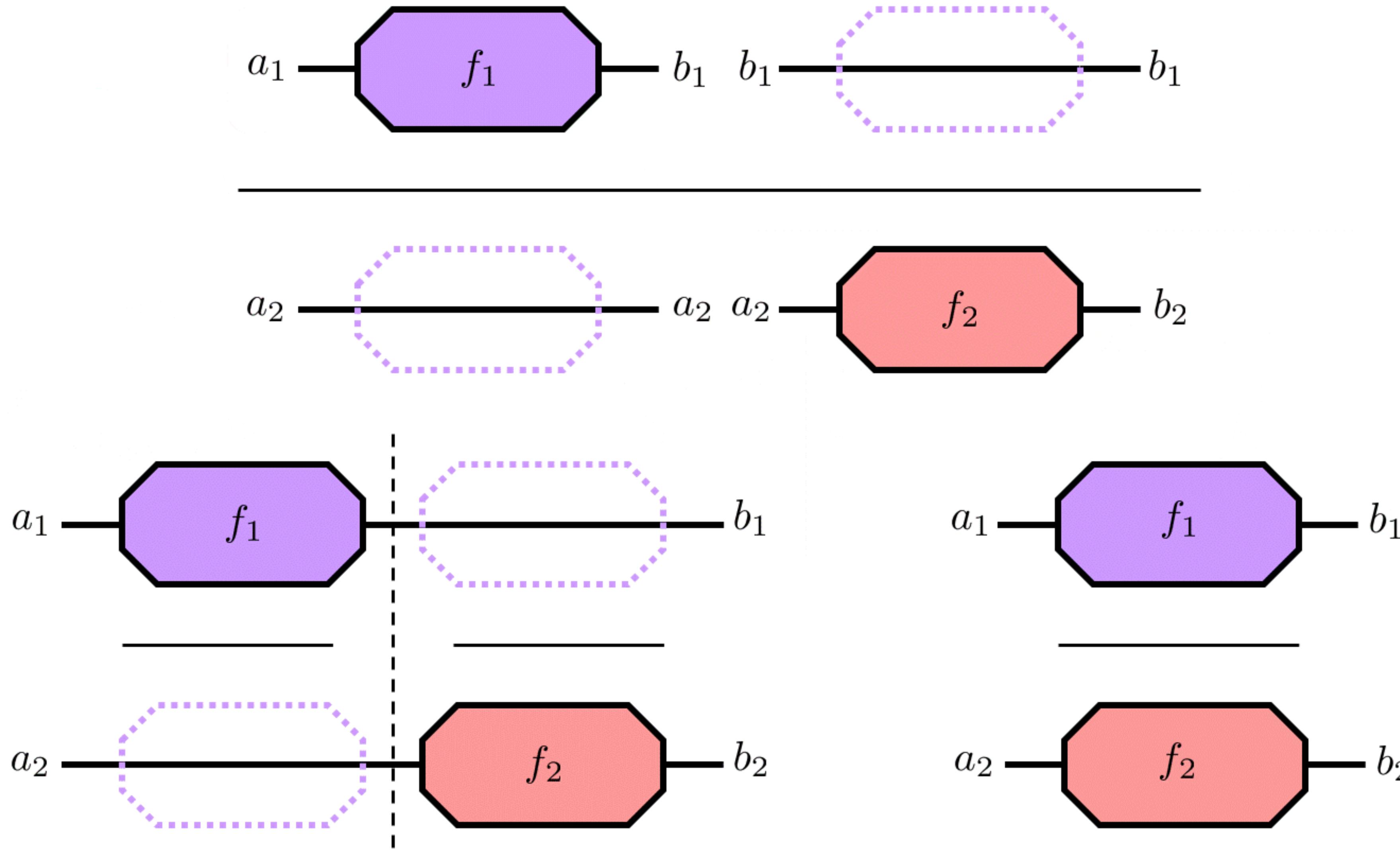
$B \mid Y$ Y



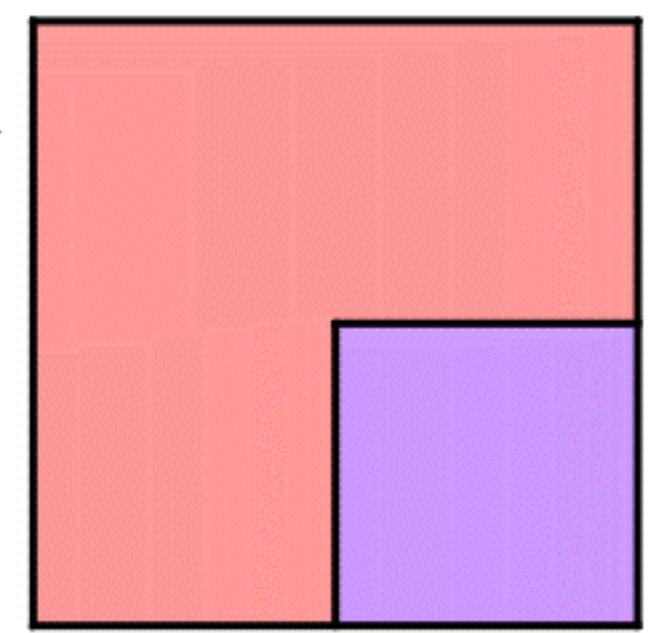
$B \mid Y$ Y



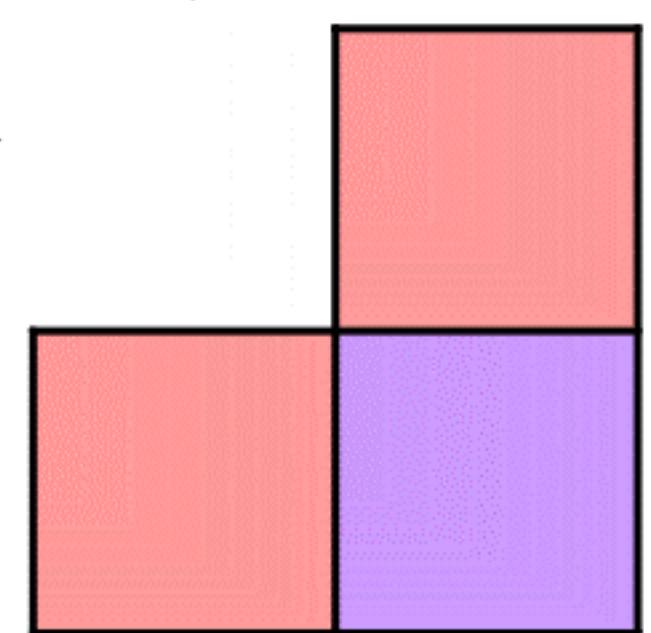
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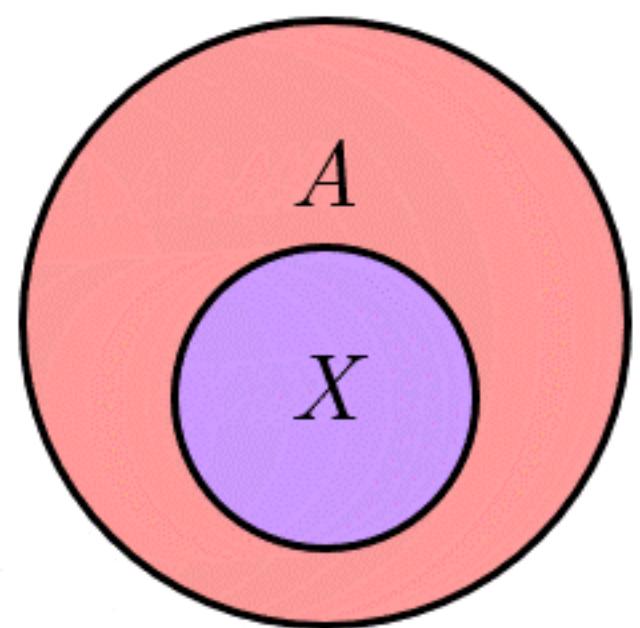
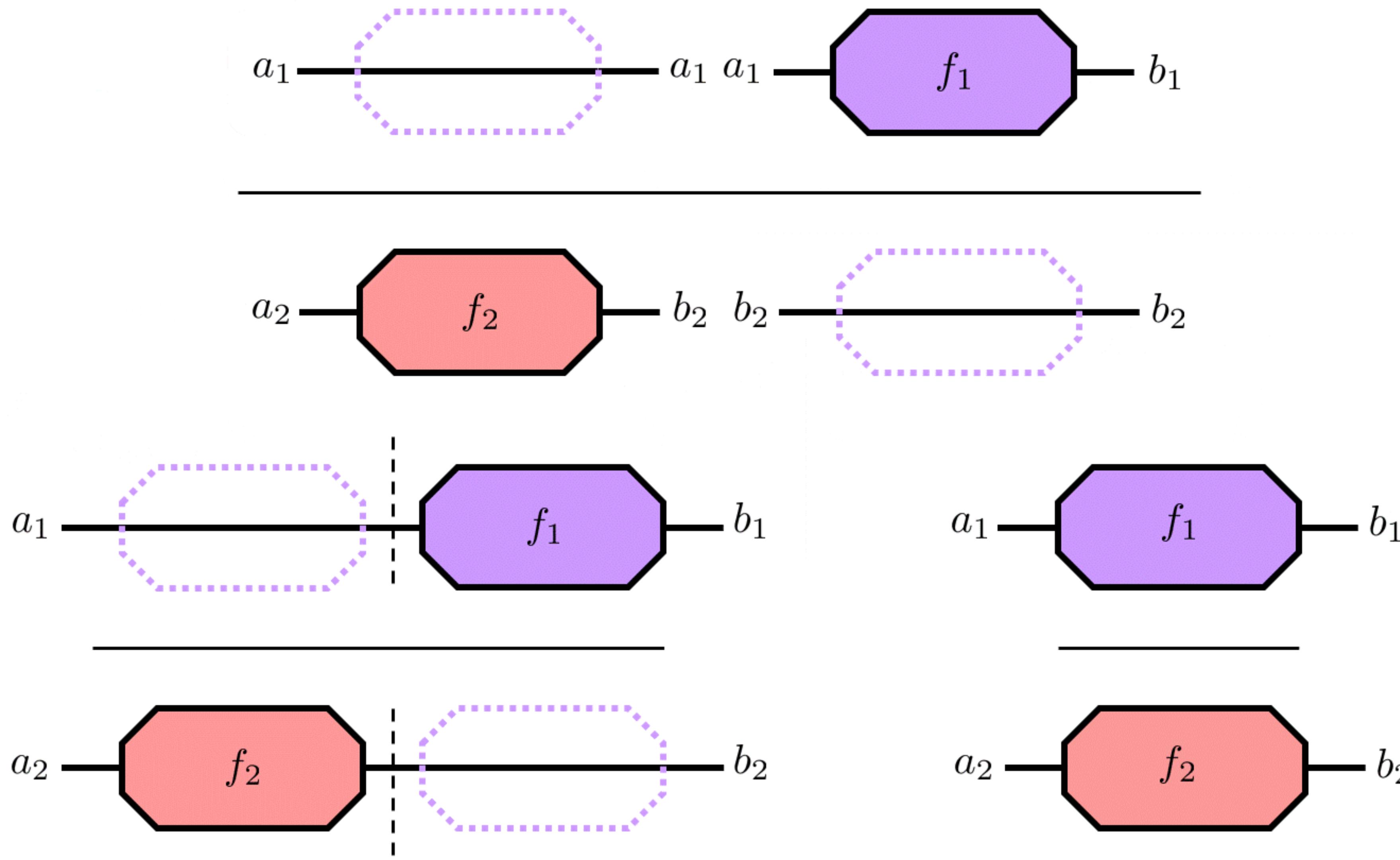
$B \mid Y$ Y



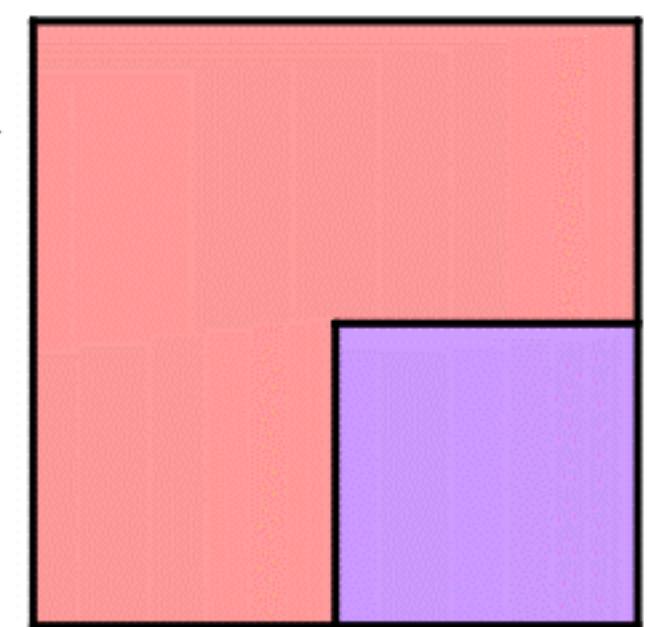
$B \mid Y$ Y



Freyd and Subset-Freyd



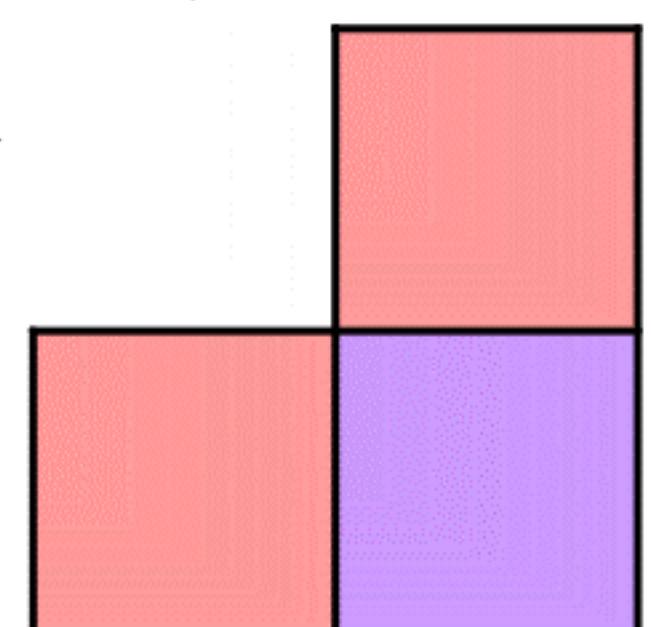
$B|Y$ Y



$A|X$

X

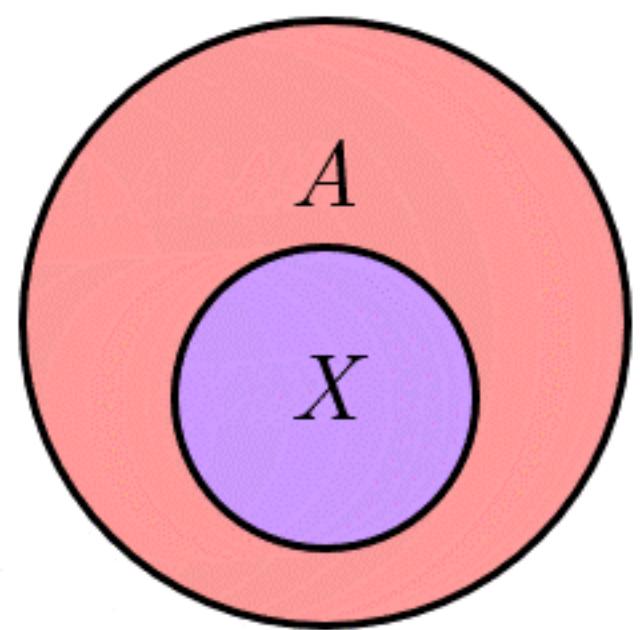
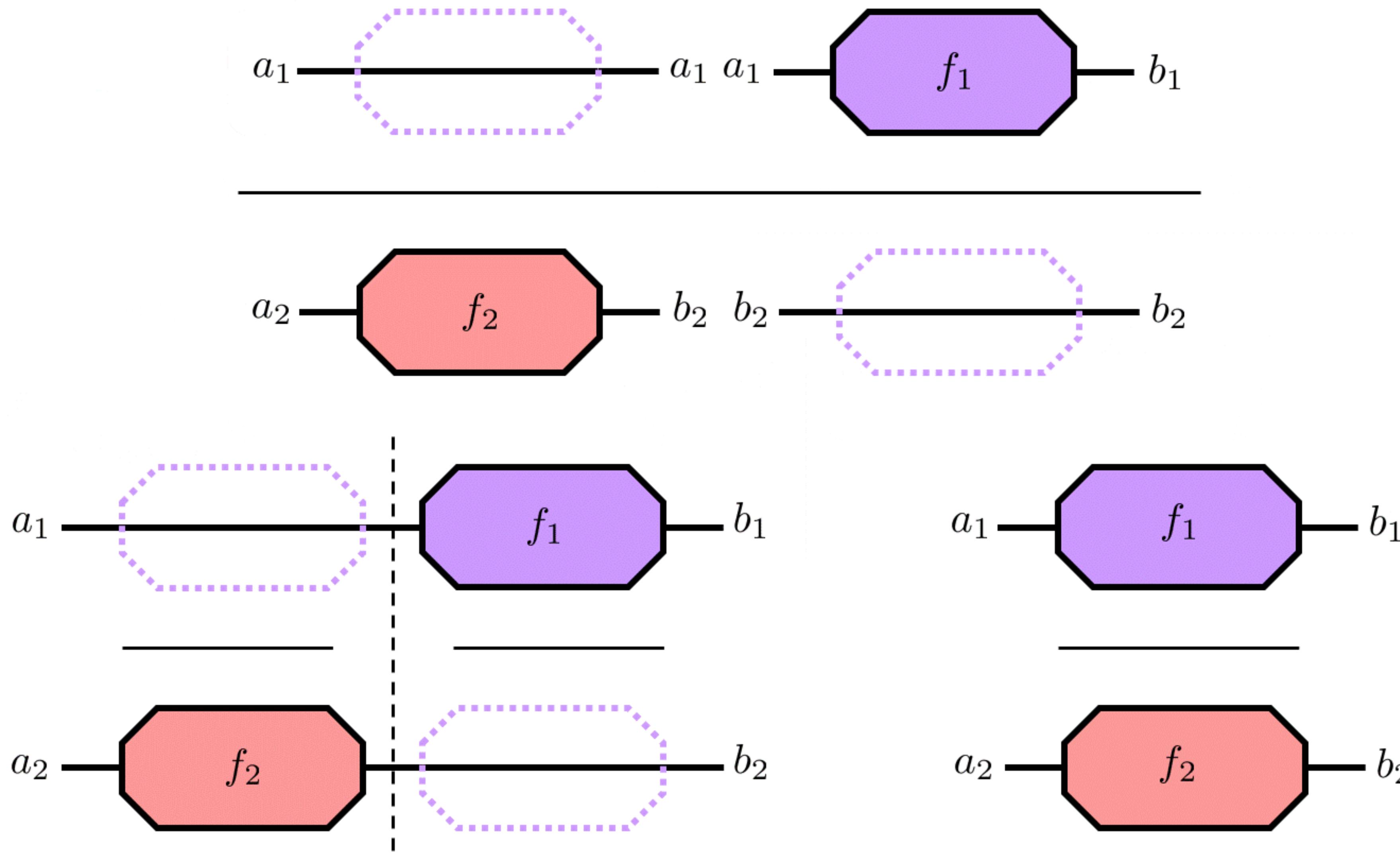
$B|Y$ Y



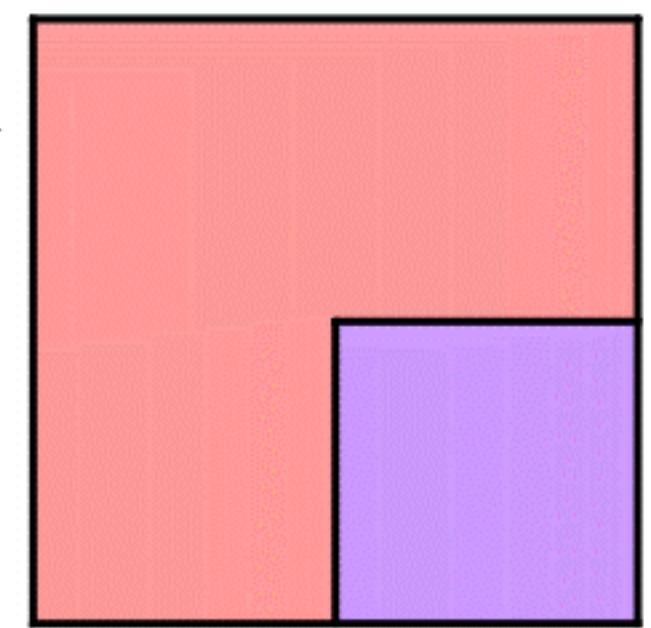
$A|X$

X

Freyd and Subset-Freyd



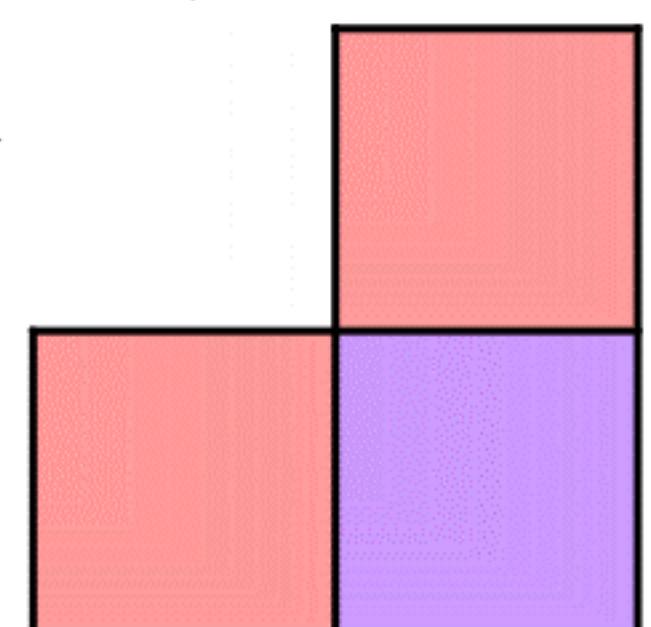
$B \mid Y$ Y



$A \mid X$

X

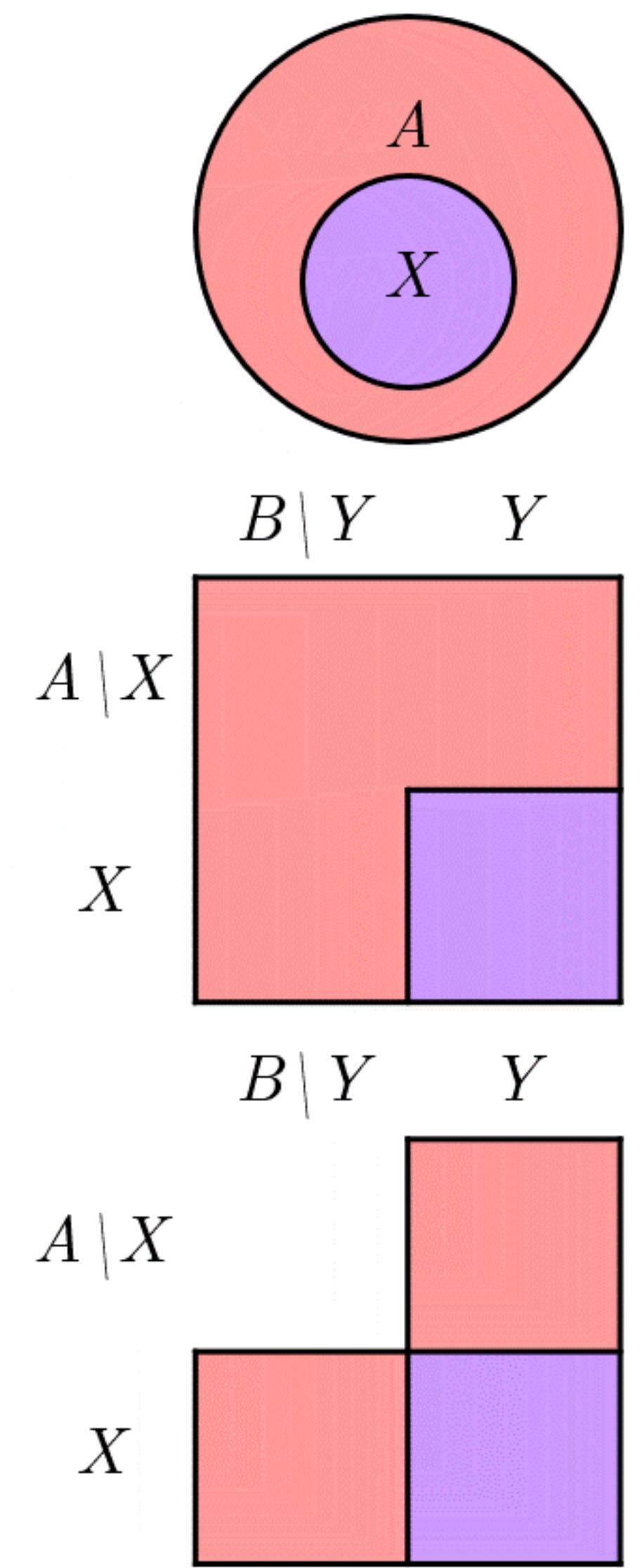
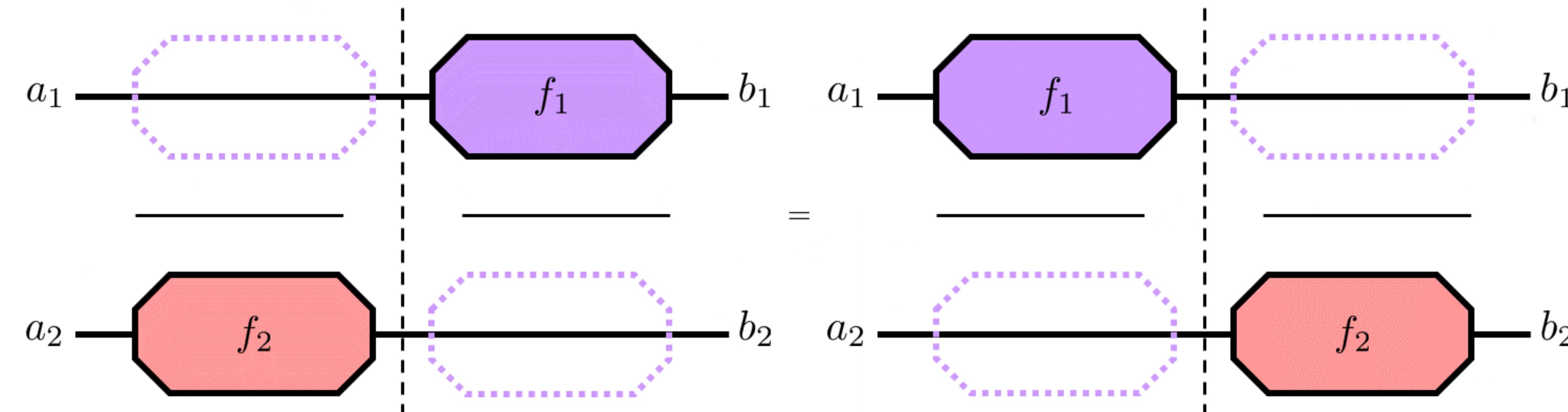
$B \mid Y$ Y



$A \mid X$

X

Freyd and Subset-Freyd

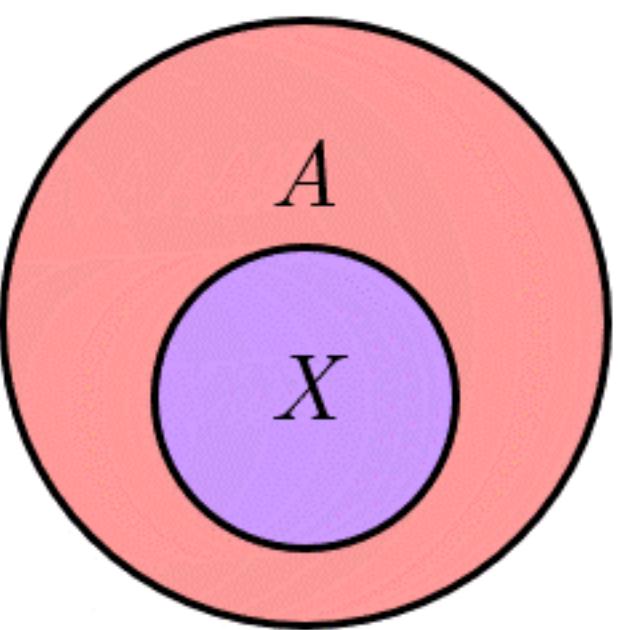


Freyd and Subset-Freyd

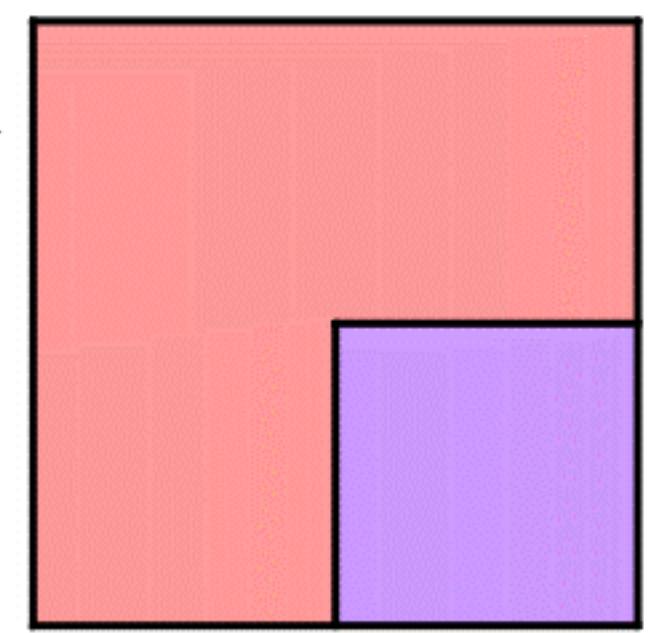


Freyd is equivalent to the full coreflective subcategory of **Subset-Freyd** s.t.

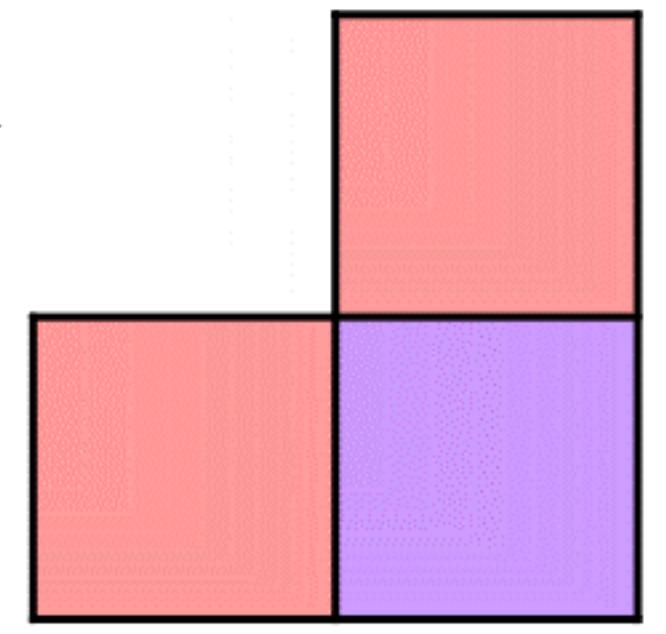
Subset-Freyd is **Freyd** with a little bit more information.



$B \mid Y$ Y



$B \mid Y$ Y



More in the paper



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- Abstract characterization of duoidally enriched Freyd categories as monoids in some category.

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- Change of enrichment and applications using Yoneda and forgetful functors.

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- More examples, indexed state and Kleisli categories for changing Lawvere theories.

Conclusion

- Summary of findings
- Implications for practice
- Limitations of study
- Future research directions

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Summary

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Further work

- Improve the abstract characterisation.
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Thanks for listening!