Duoidally Enriched Freyd Categories

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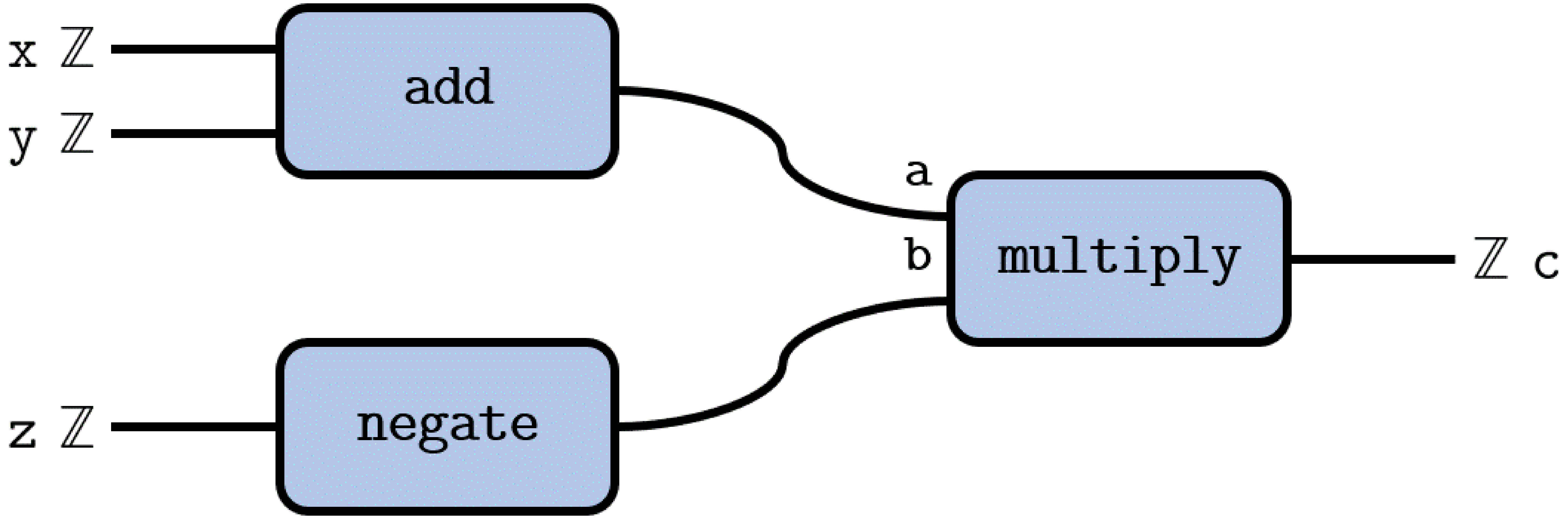
Roadmap

- Motivation
- Background definitions
- Duoidally enriched Freyd categories
- Examples
- More from the paper

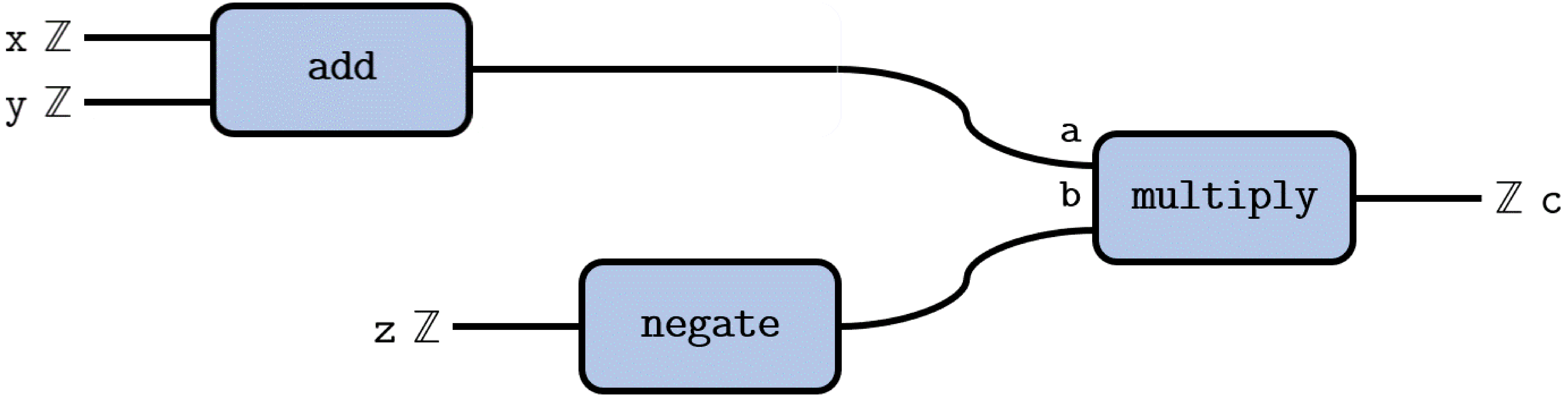
Motivation

Motivation




```
let (a, b) = (add(x, y), negate(z)) in
let      c = multiply(a, b) in
c
```

Motivation

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let a = add(x, y) in
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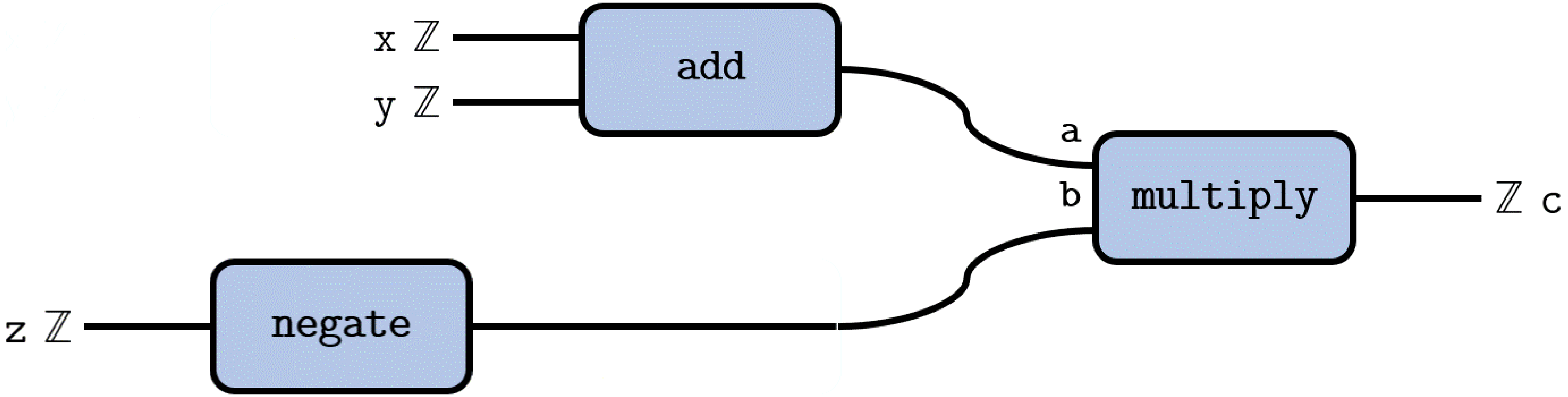


```
let b = negate(z) in
let a = add(x, y) in
let c = multiply(a, b) in
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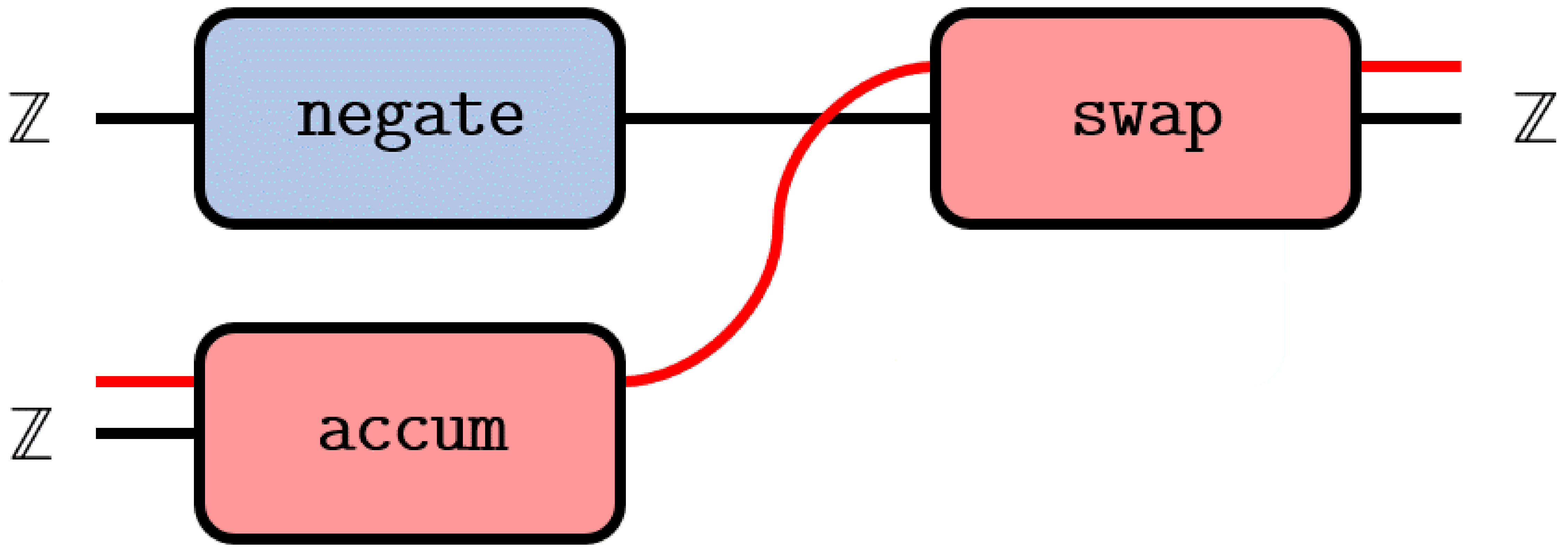
```

Motivation



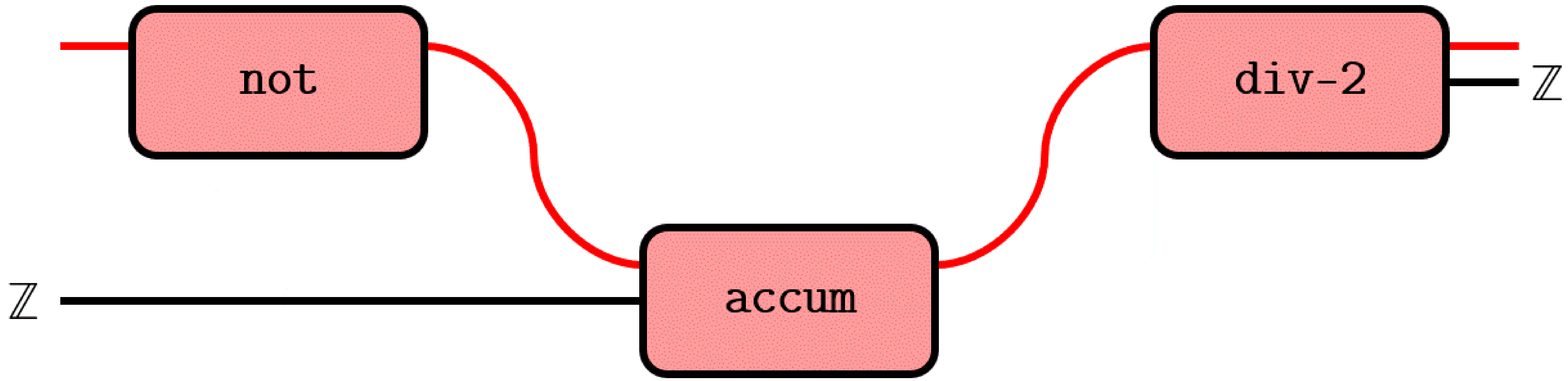

```
negate : ℤ -> ℤ
negate(x) = -x

accum : ℤ -> ()
accum(y) =
  let s = get() in
  put(y + s)

swap : ℤ -> ℤ
swap(z) =
  let s = get() in
  put(z);
  s
```

effect-free effectful

Motivation



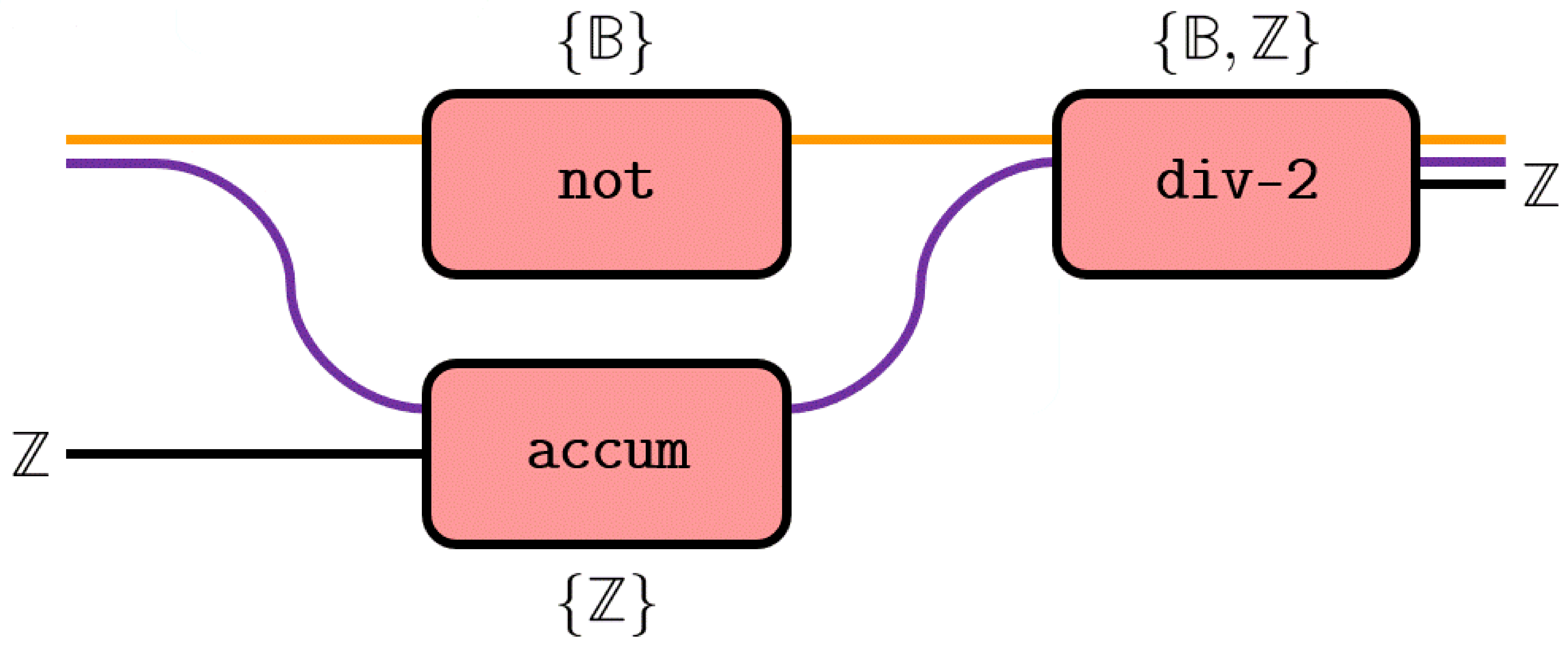
effectful not parallel

```
not : () -> ()
not() =
  let (b, s) = get() in
  put(¬b, s)

accum : ℤ -> ()
accum(y) =
  let (b, s) = get() in
  put(b, y + s)

div-2 : () -> ℤ
div-2() =
  let (b, s) = get() in
  if b then
    ceil(s/2)
  else
    floor(s/2)
```

Motivation




```
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not() =
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accum : ℤ -> ()
accum(y) =
  let (b, s) = get() in
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div-2 : () -> ℤ
div-2() =
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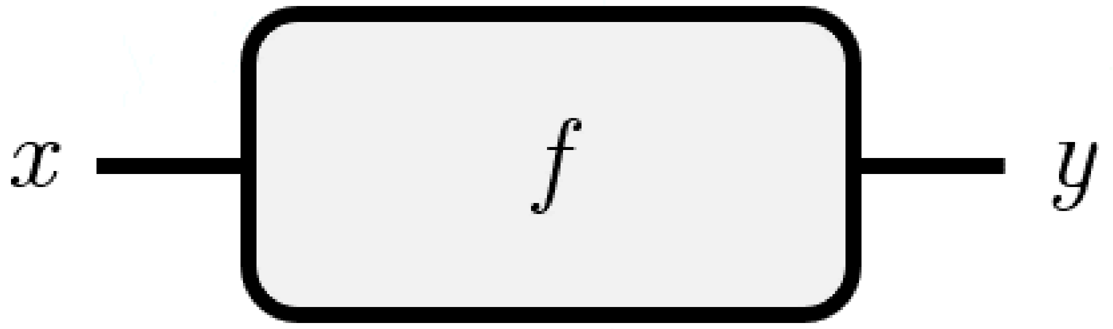
effectful in parallel

Binoidal Categories

Binoidal Categories



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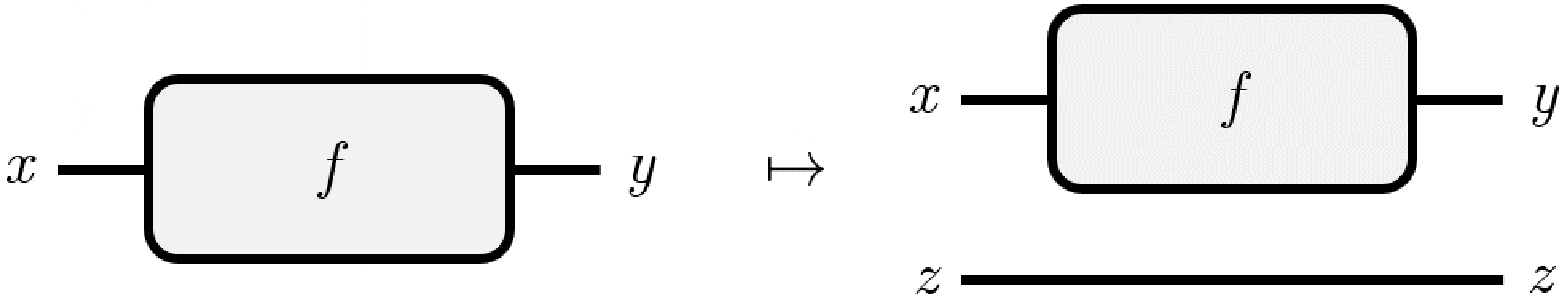
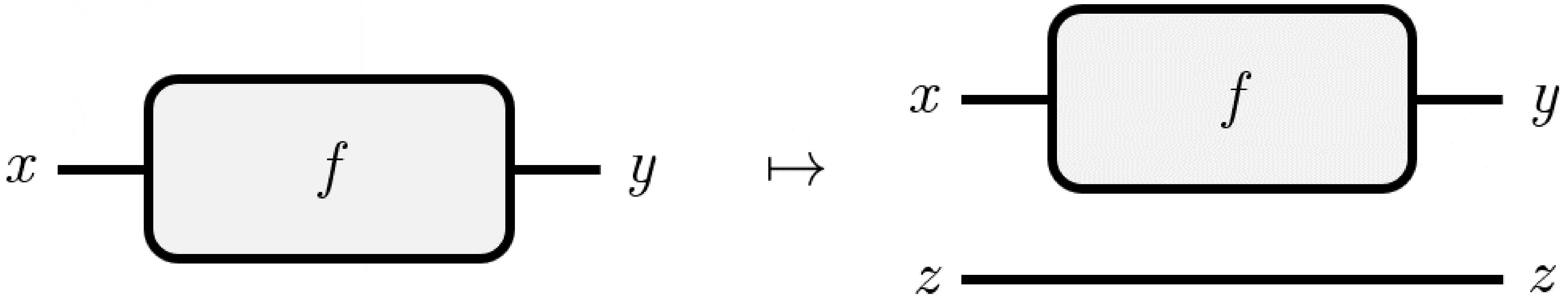


$$f \mapsto f \ltimes z$$

Binoidal Categories



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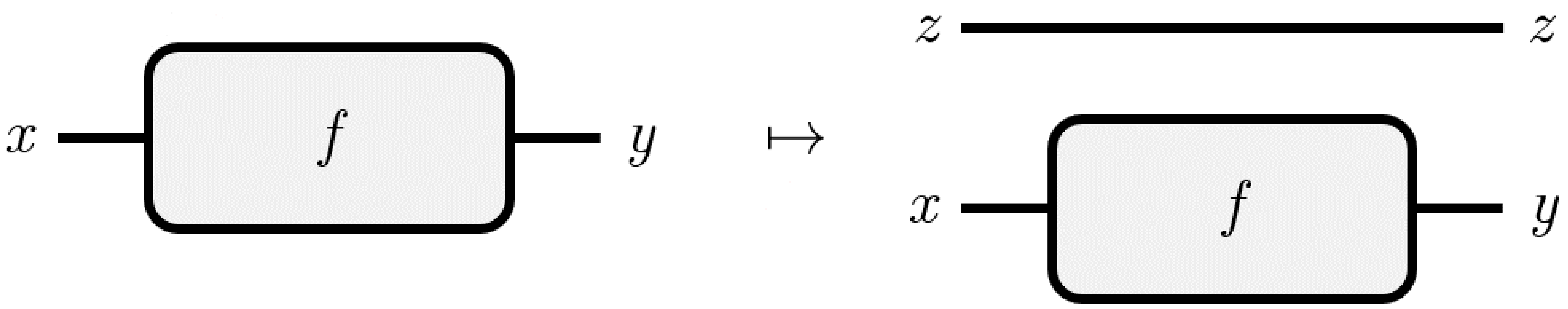
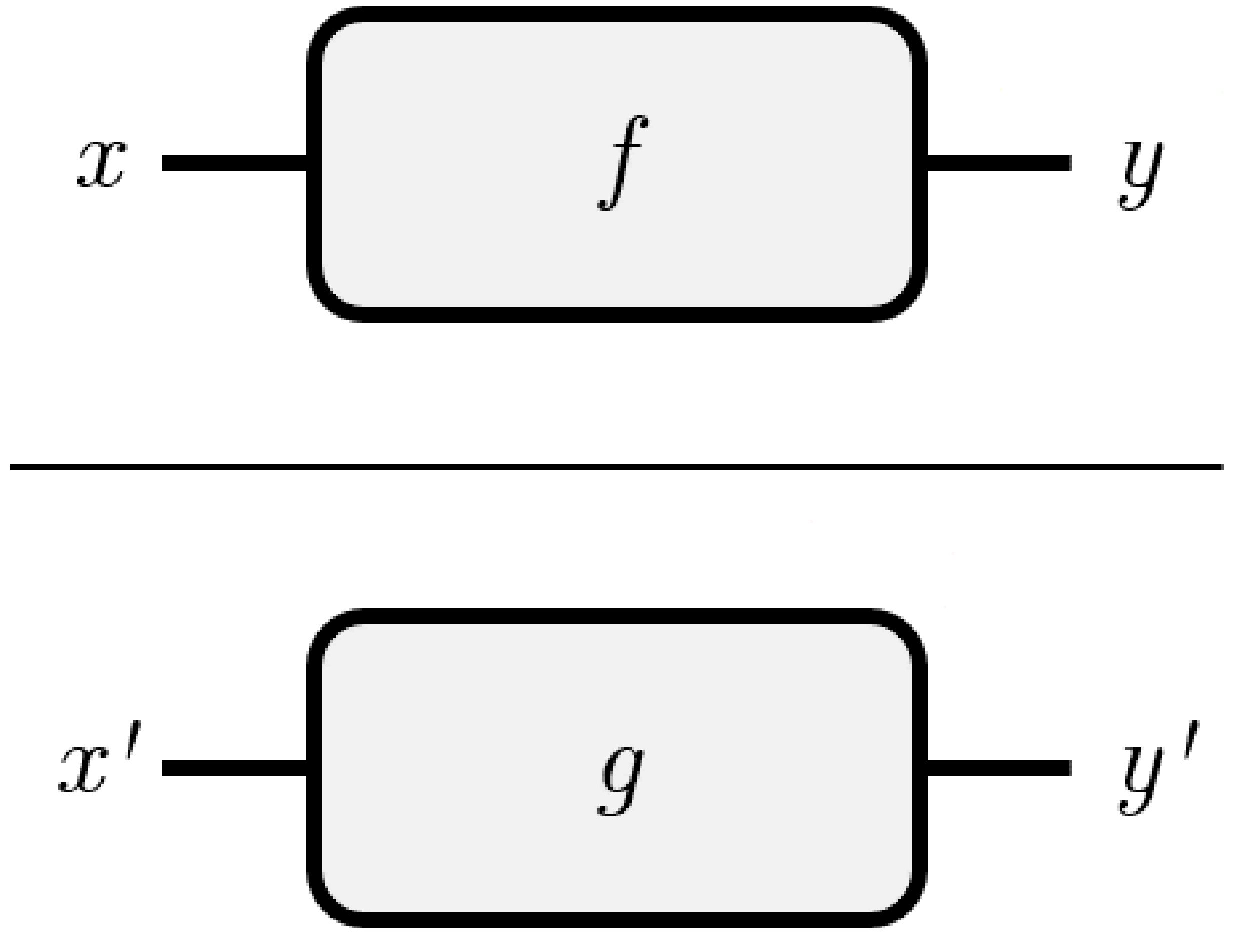
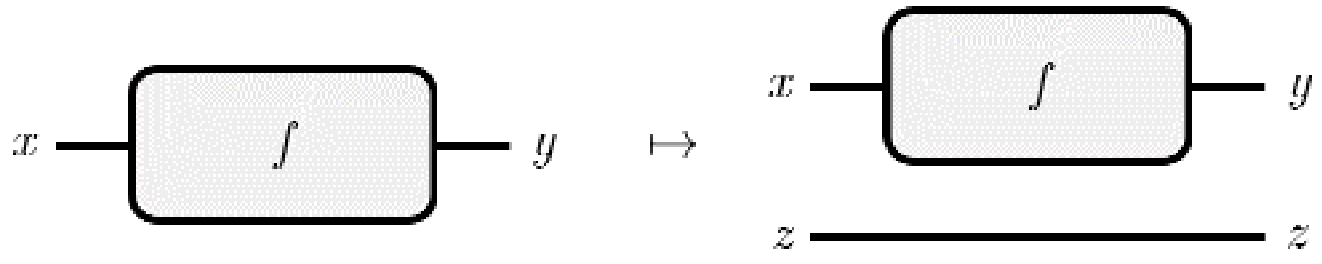


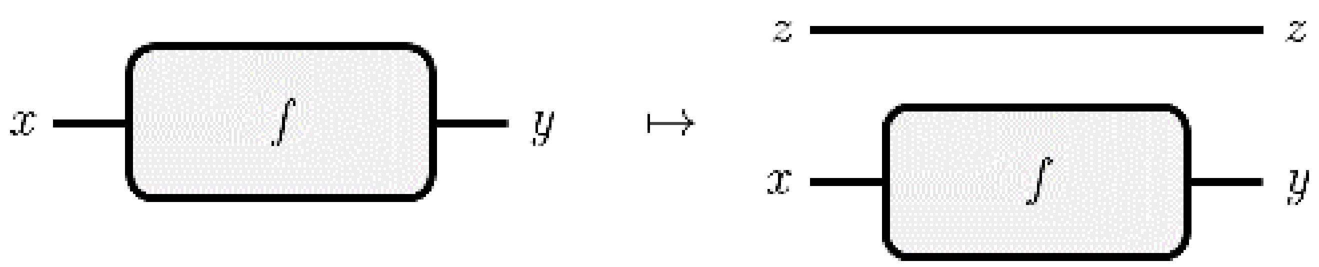
$$f \mapsto z \rtimes f$$

Binoidal Categories





$f \mapsto f \ltimes z$



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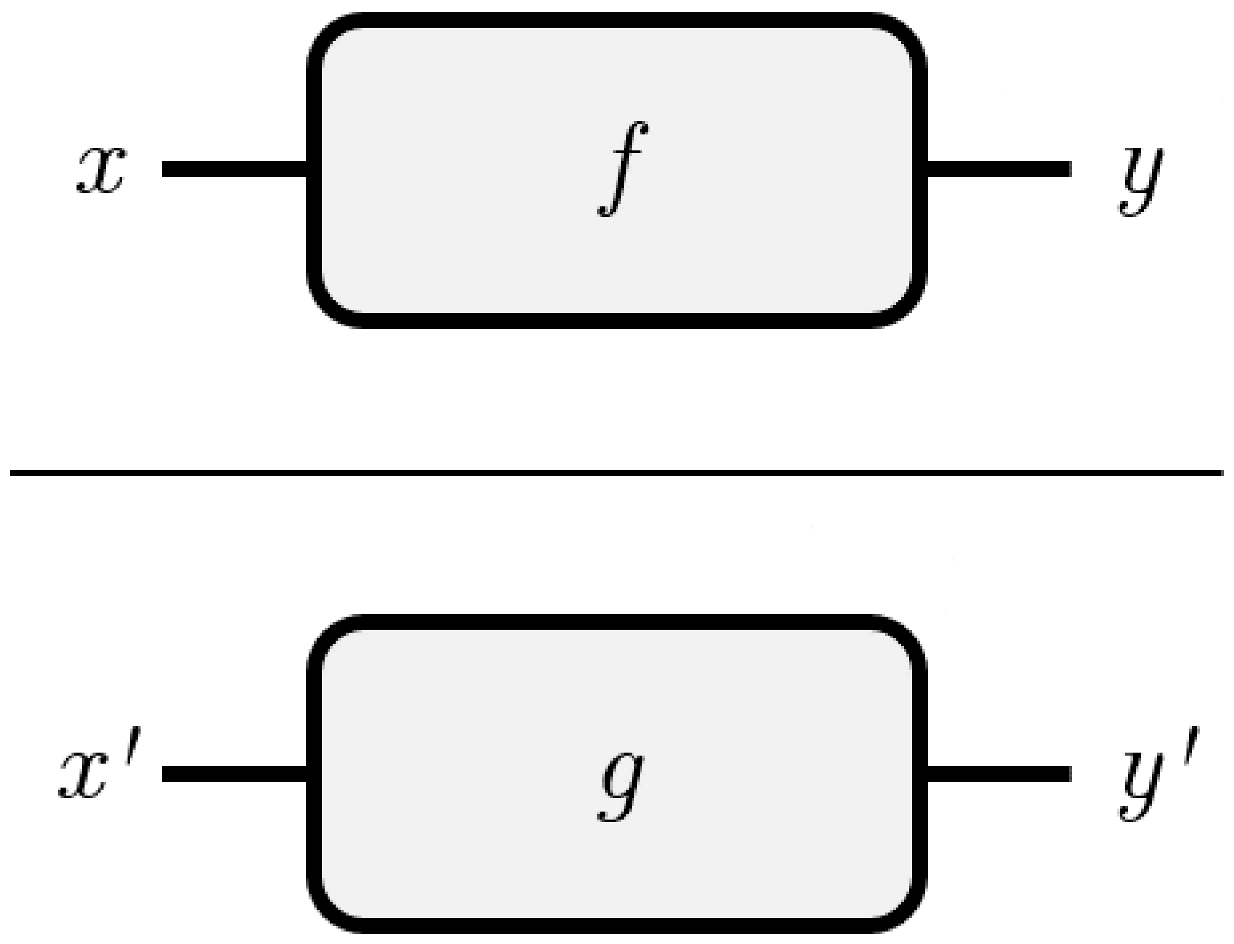

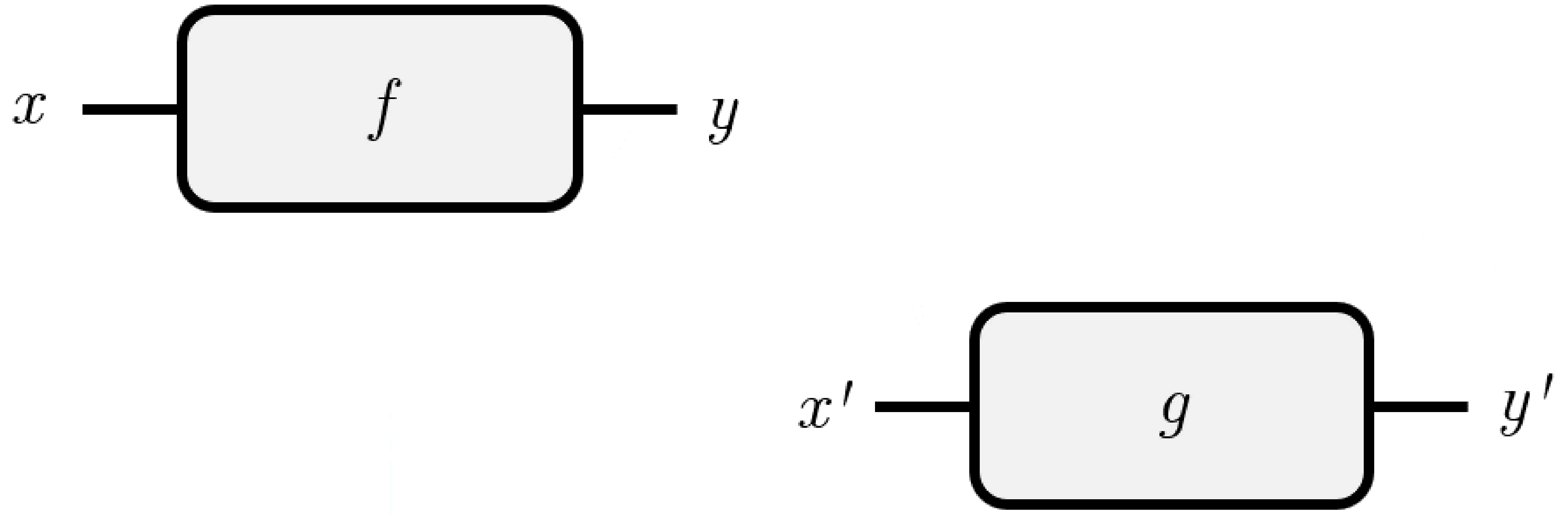

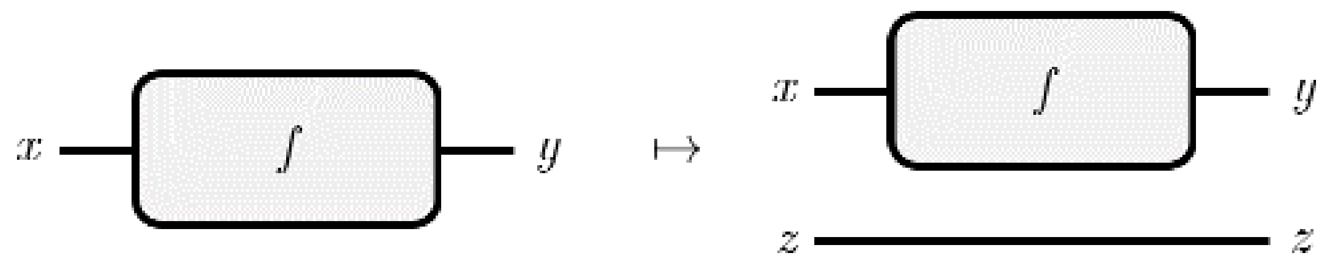

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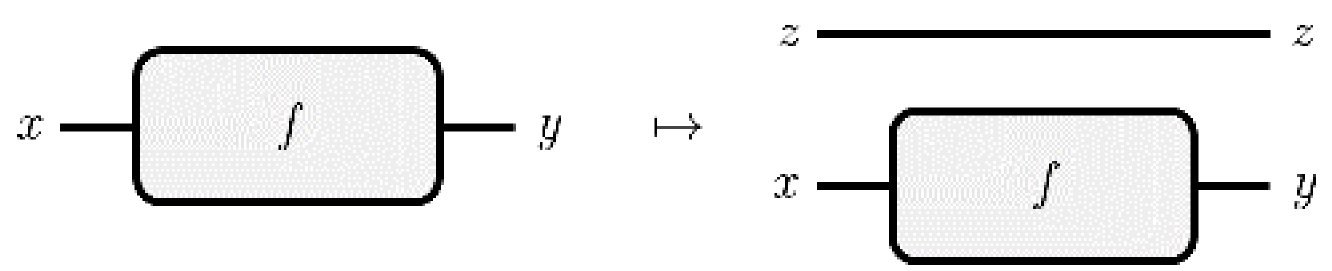

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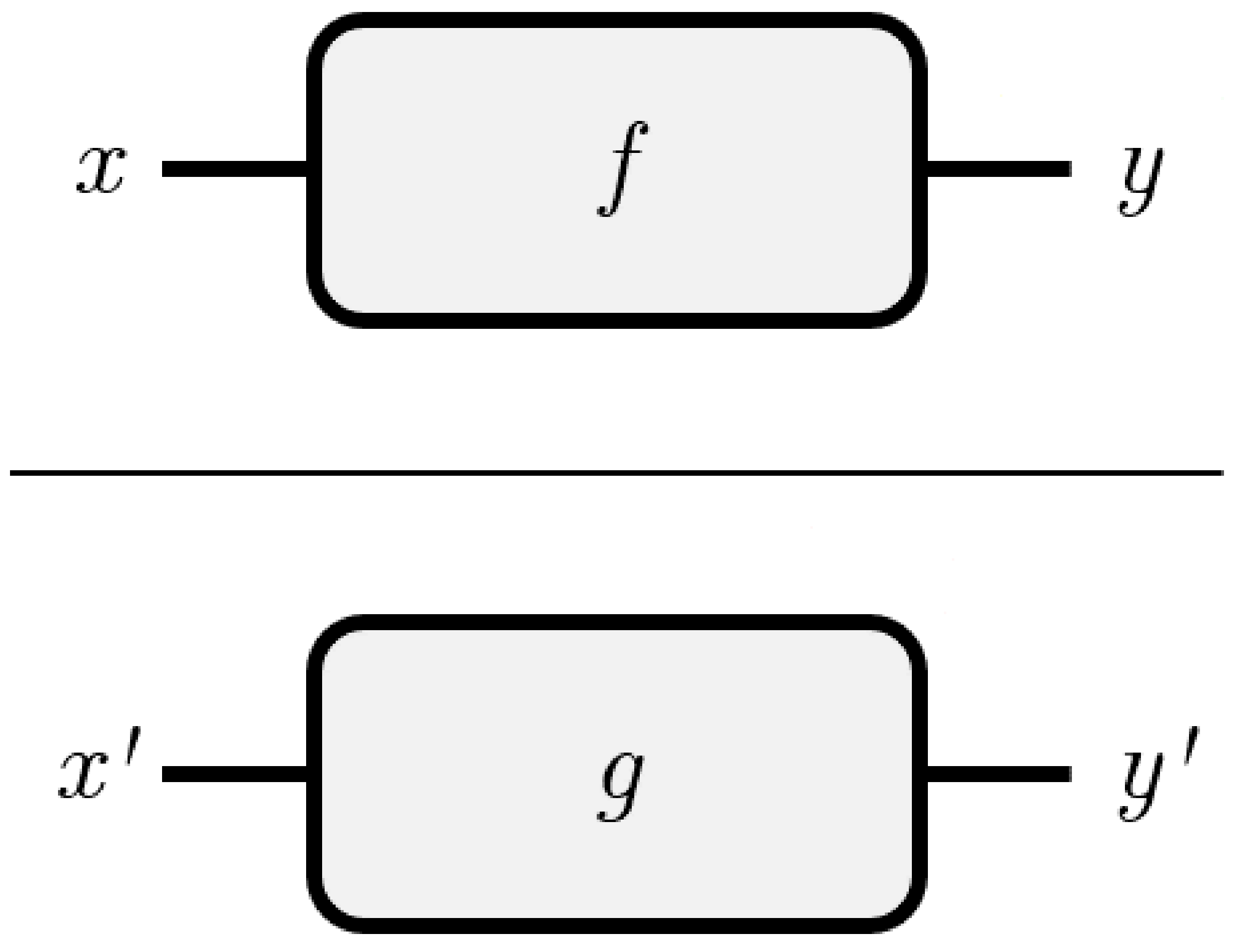

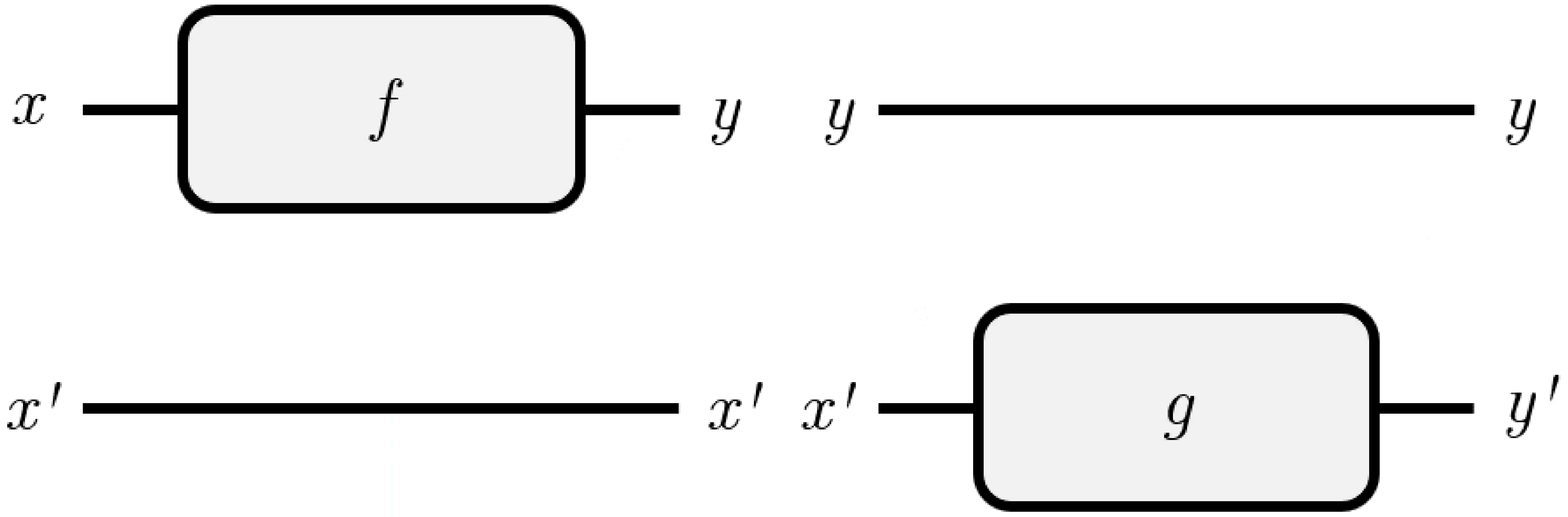

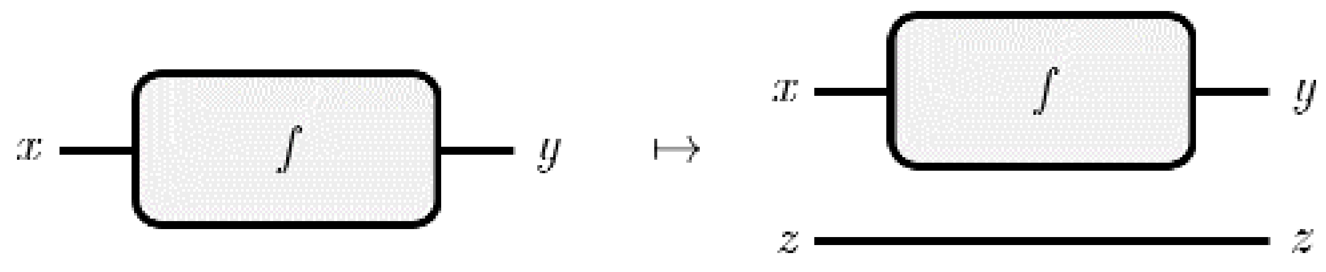

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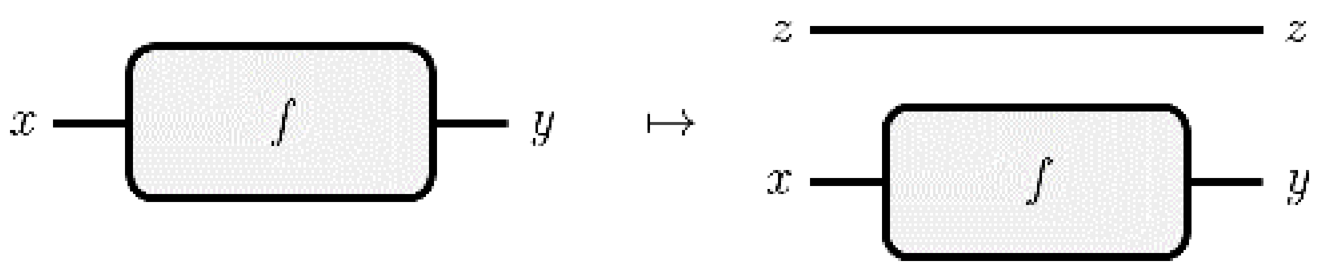

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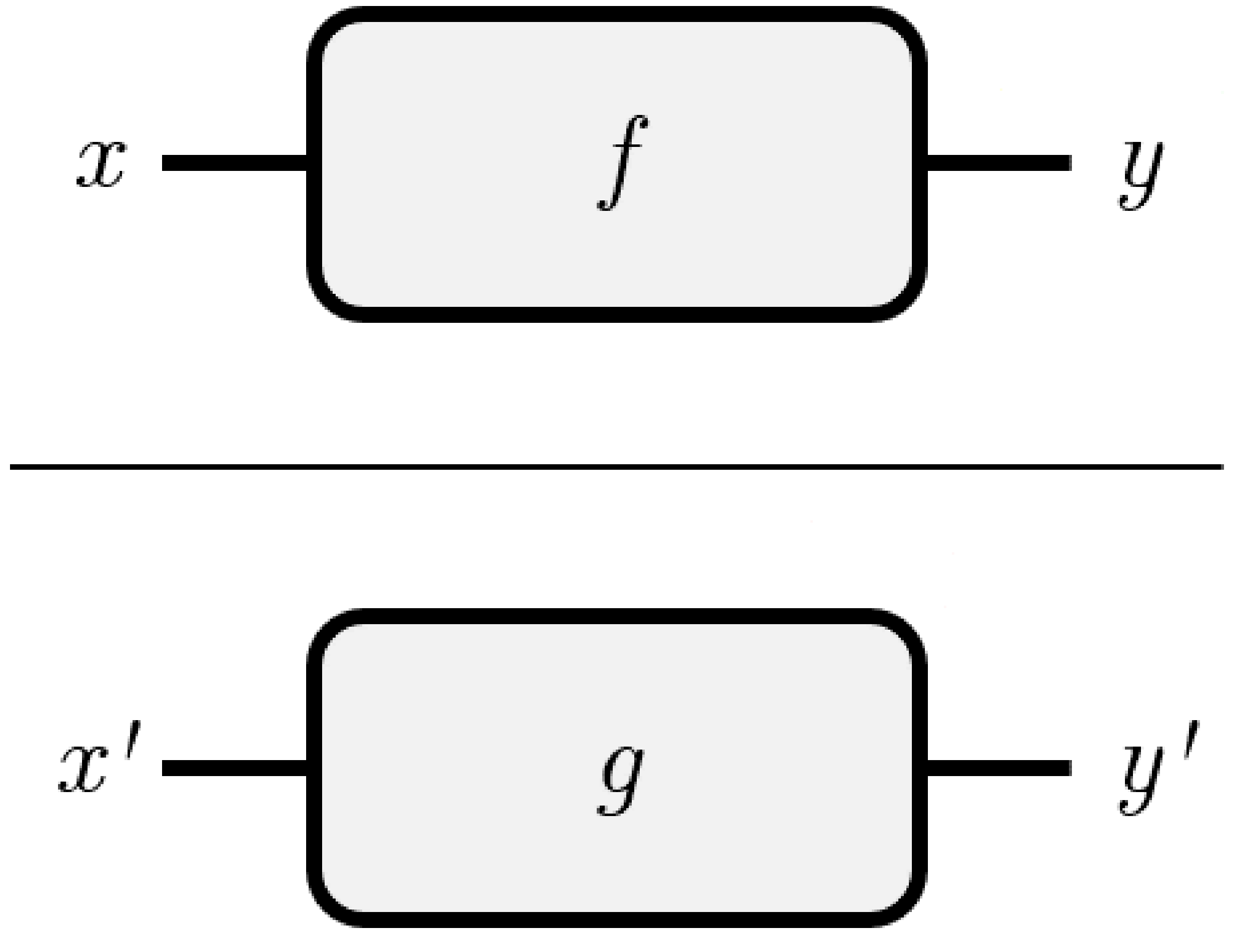
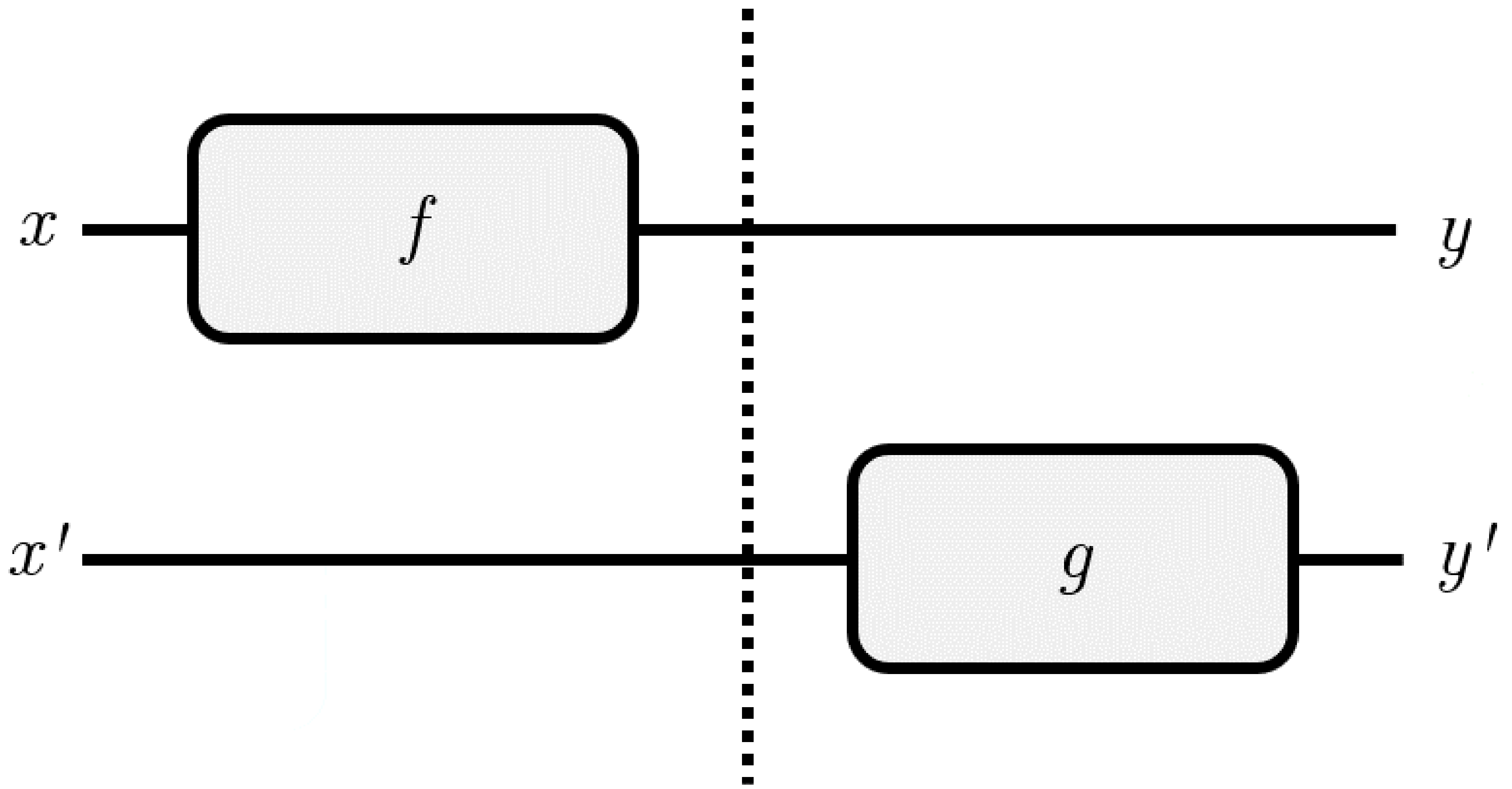
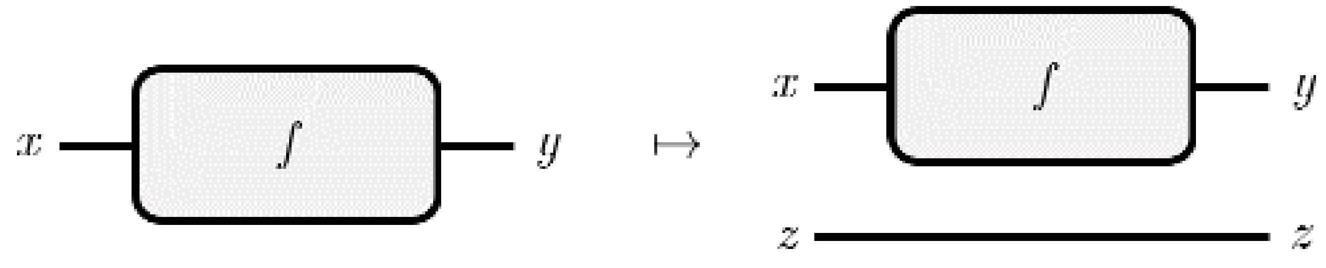




$(y \rtimes g).(f \ltimes x')$



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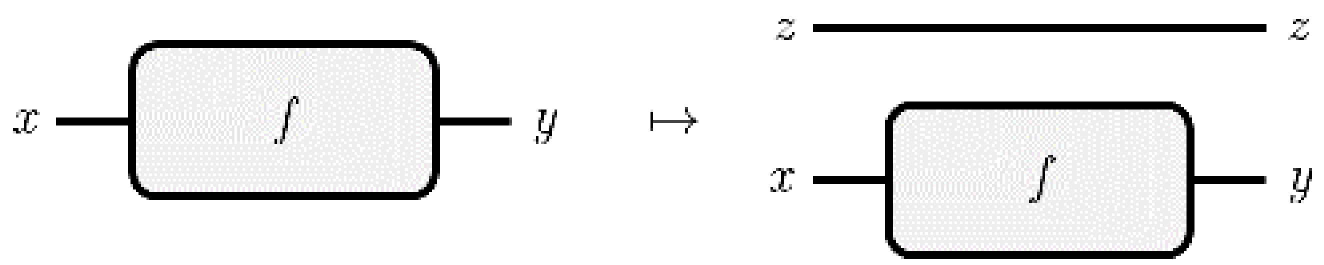

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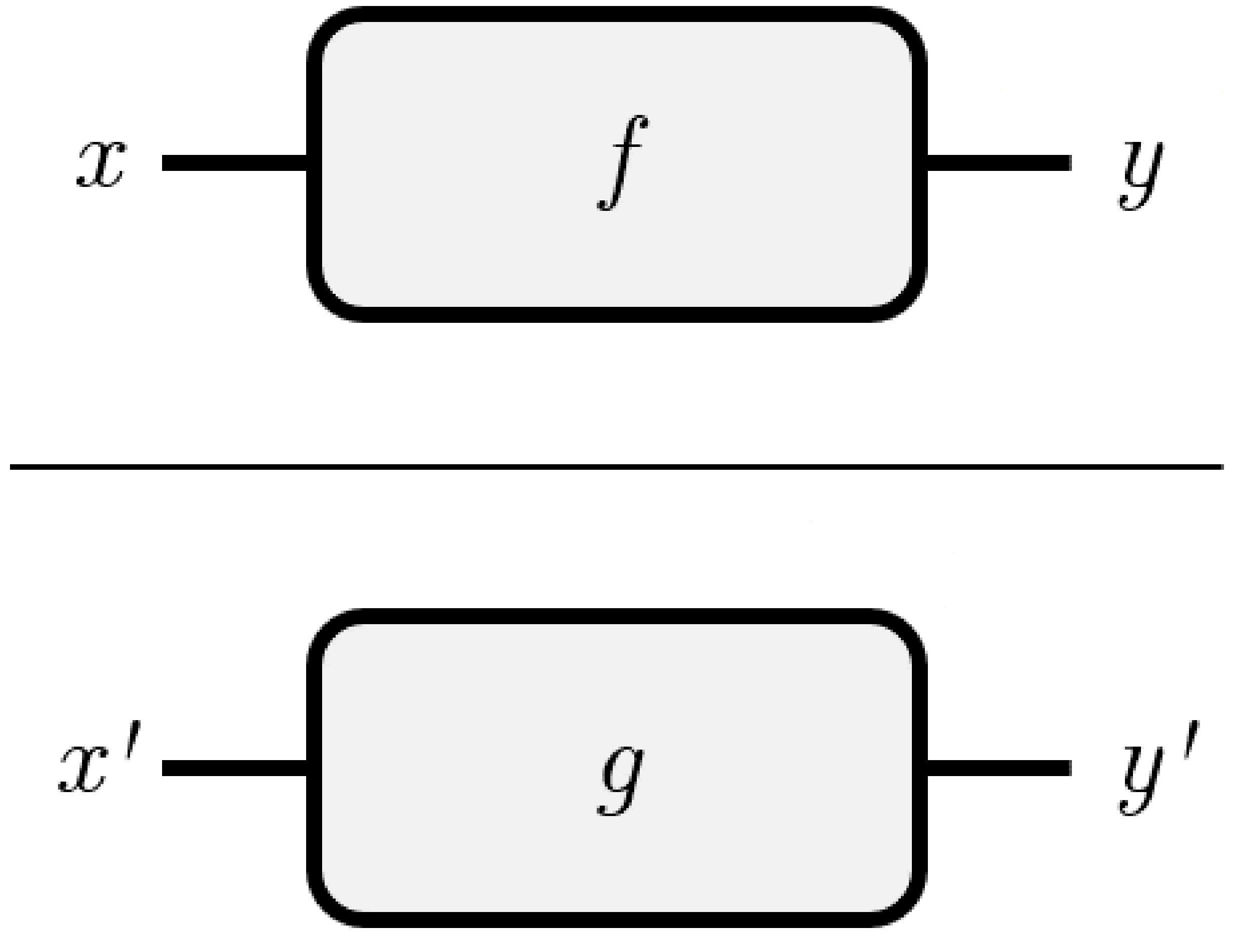
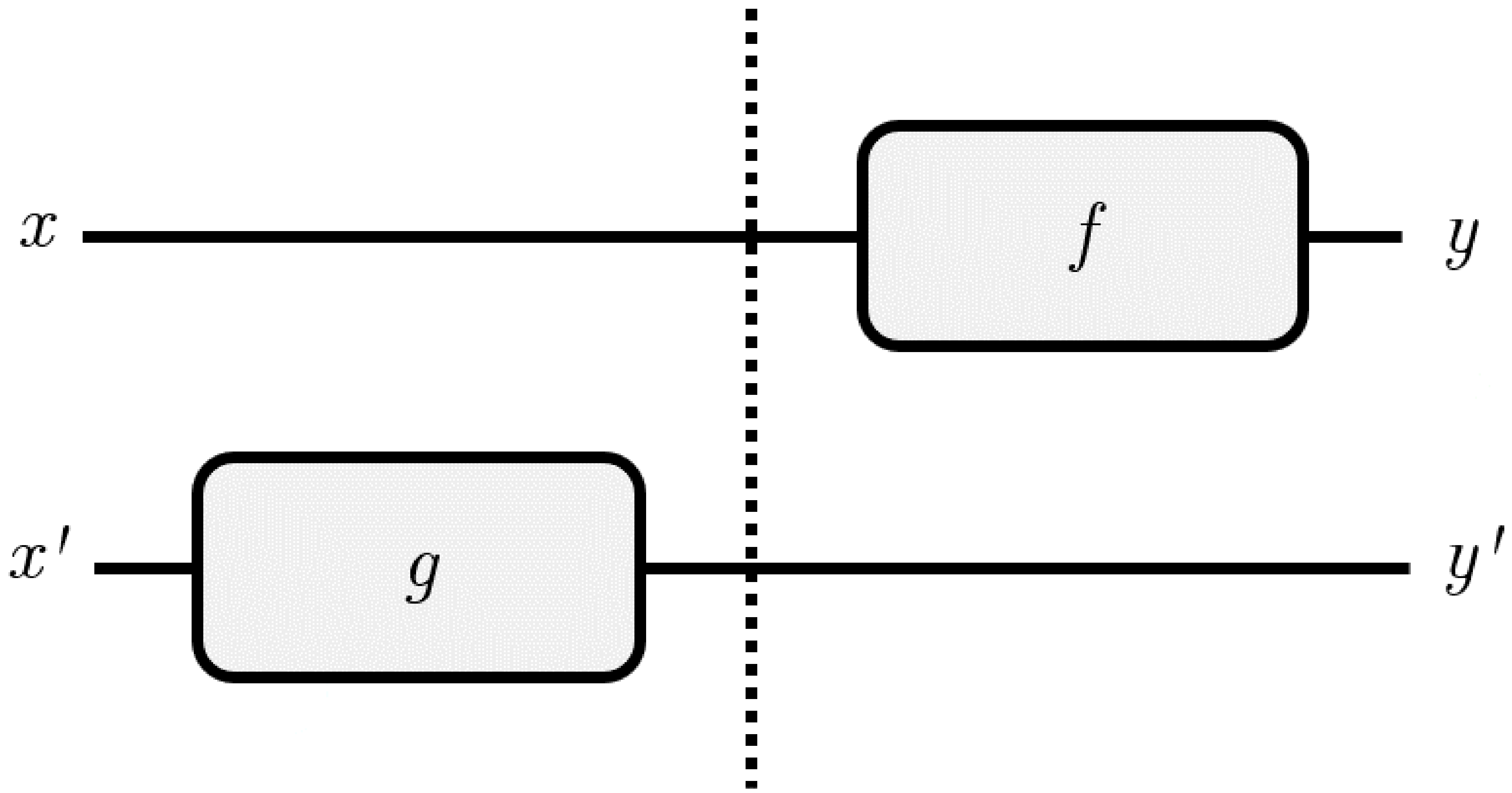
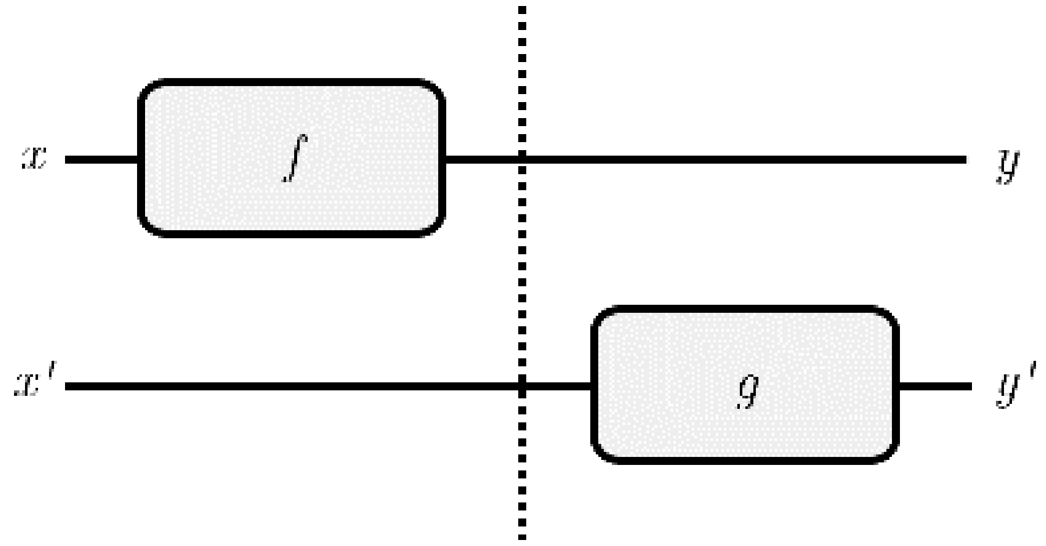
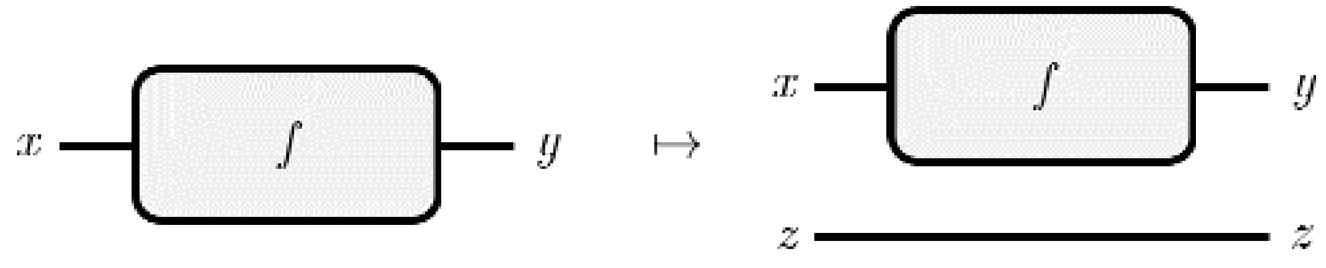




$(f \ltimes y').(x \rtimes g)$





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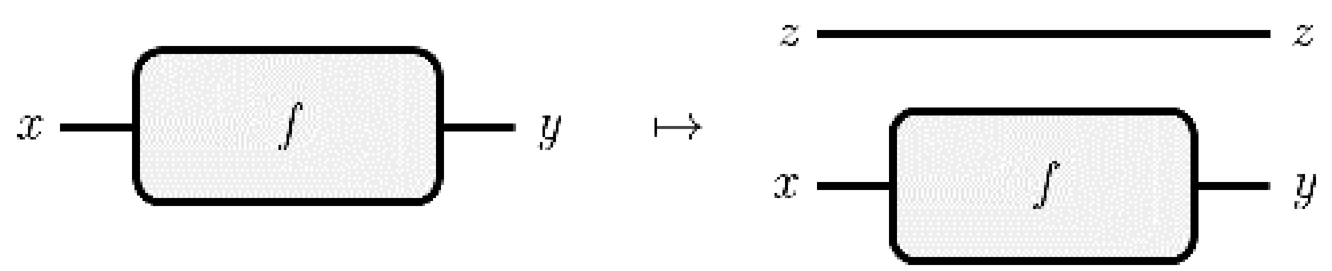

$f \mapsto z \rtimes f$

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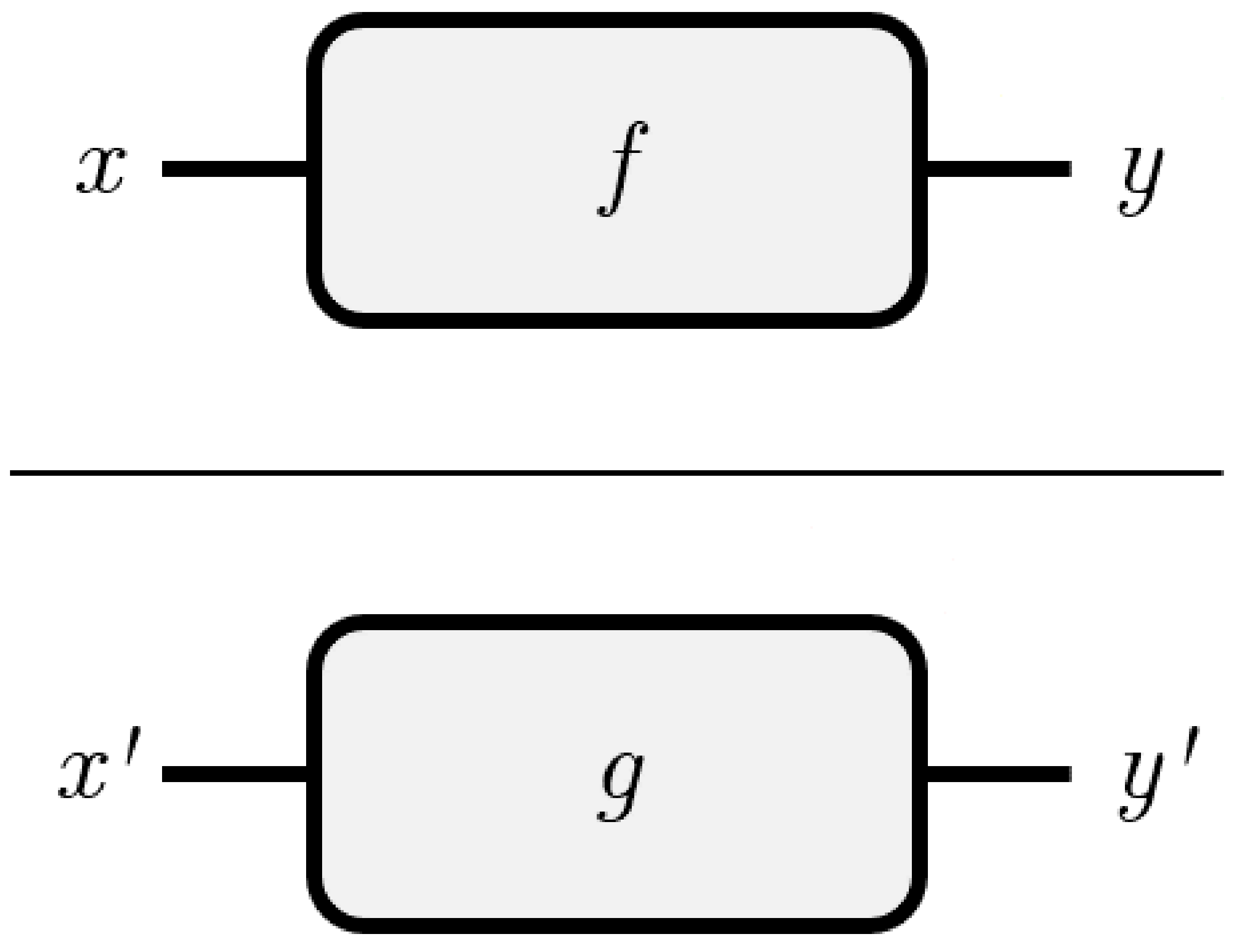
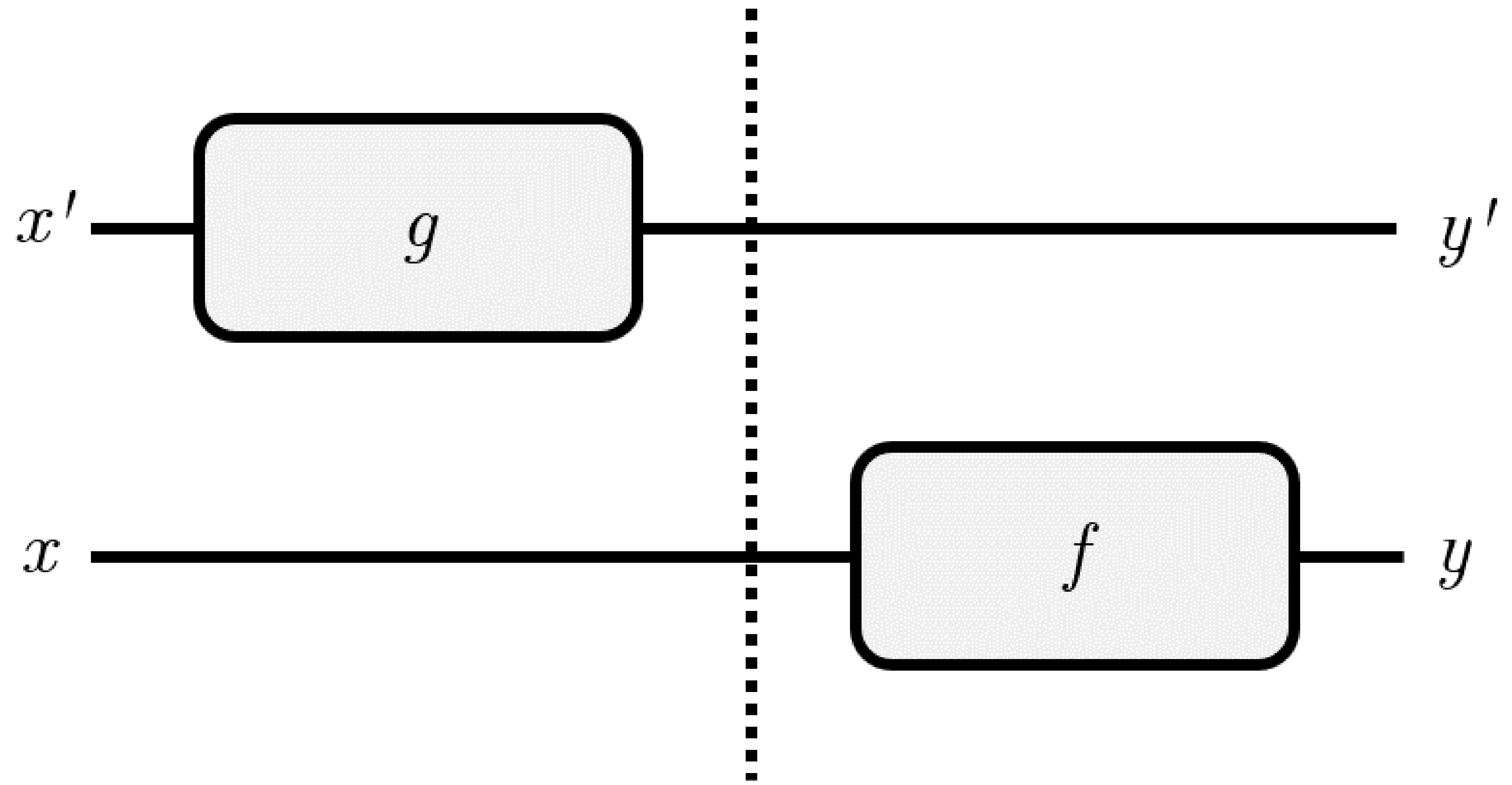
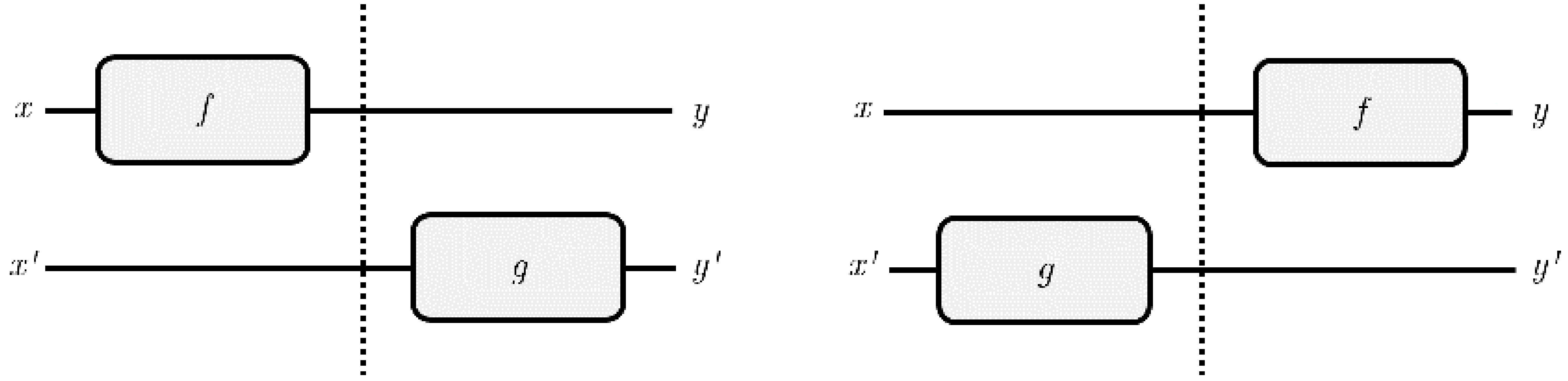
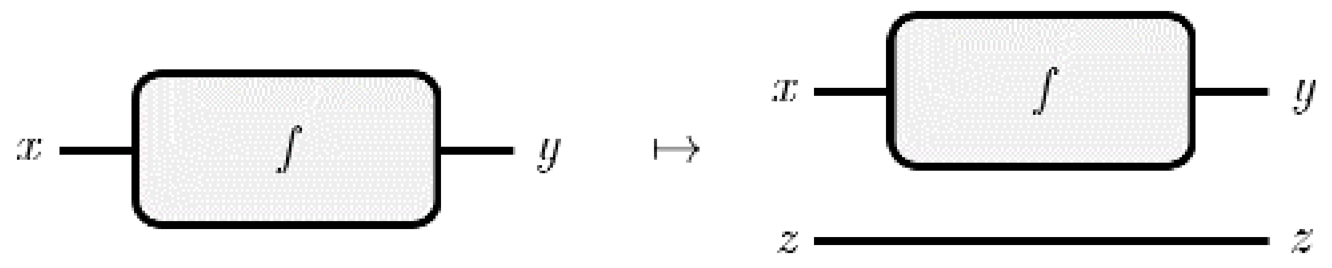




$(y' \rtimes f).(g \ltimes x)$





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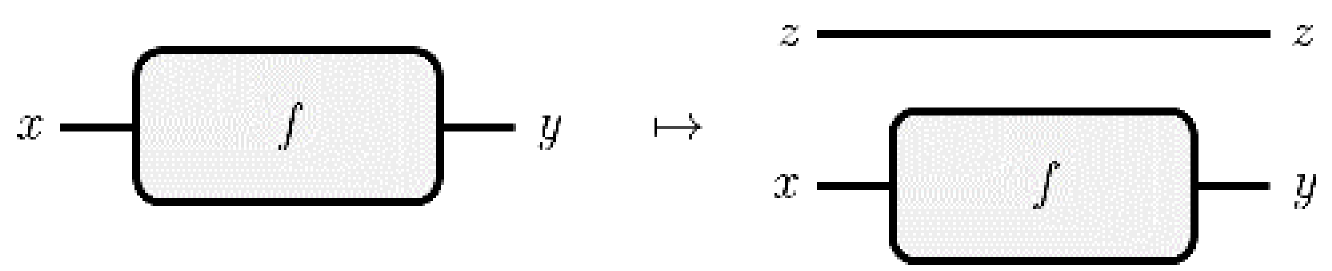

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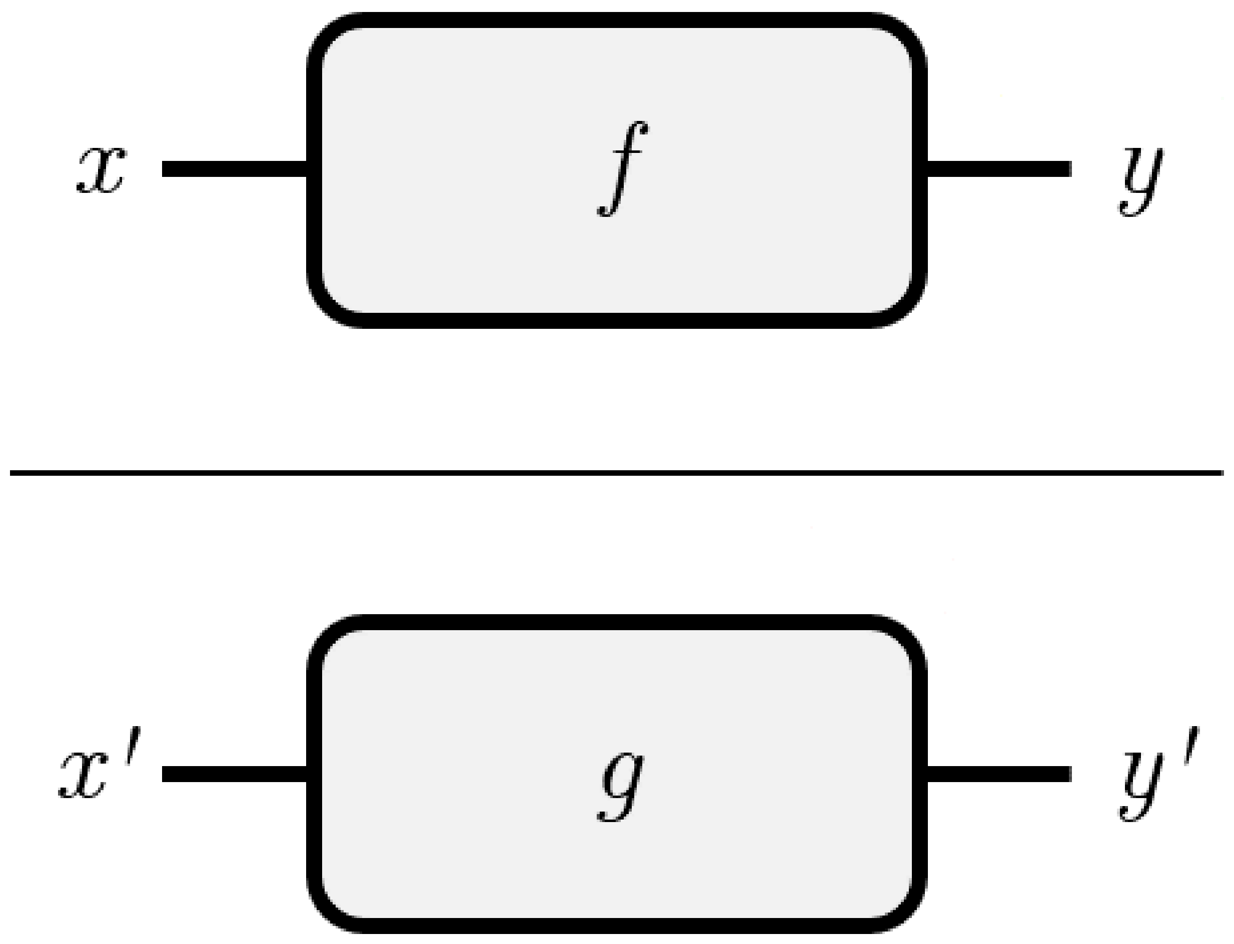
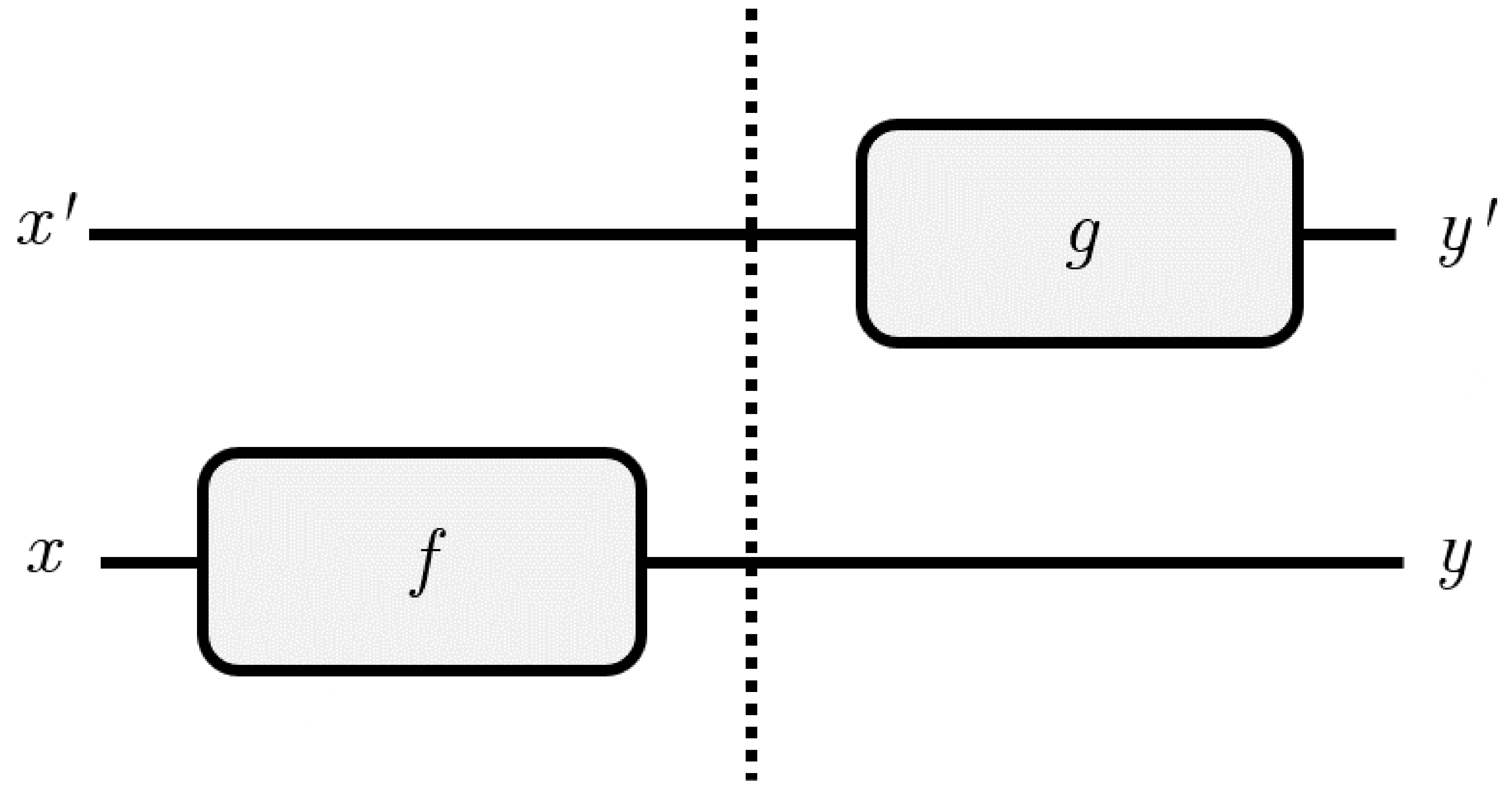
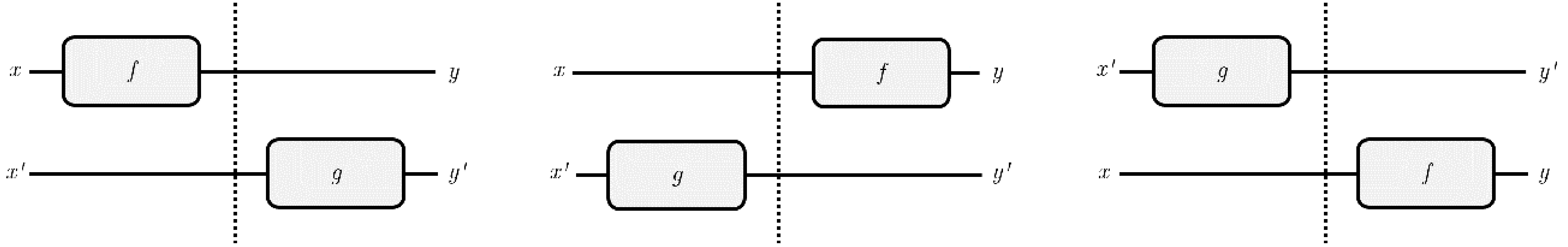
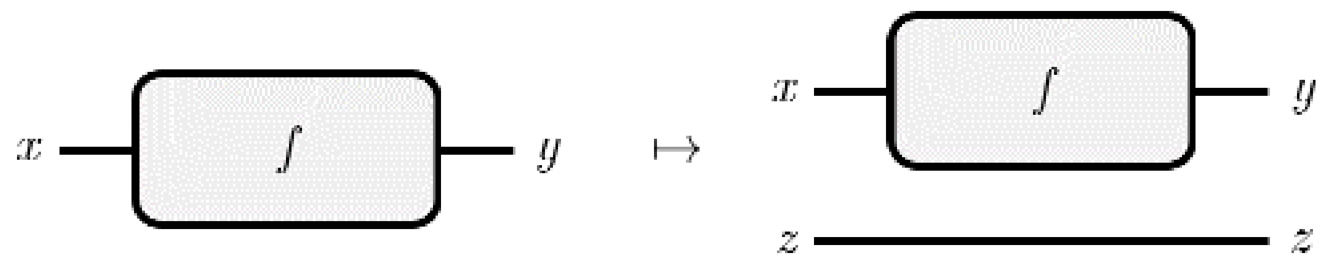




$(g \ltimes y).(x' \rtimes f)$





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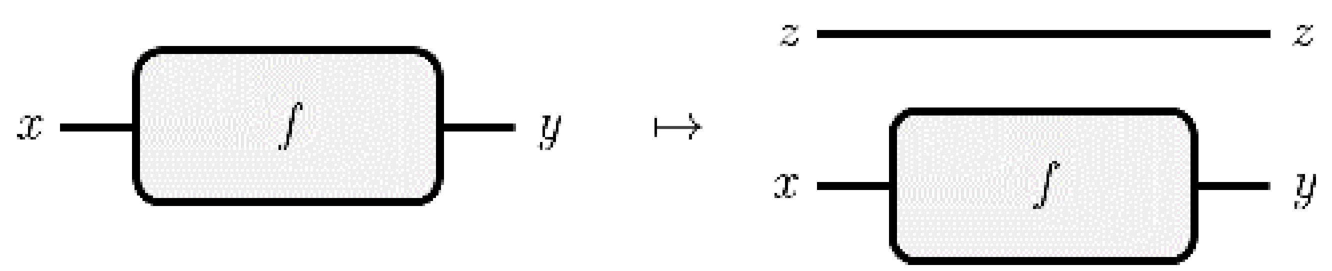

$f \mapsto z \rtimes f$

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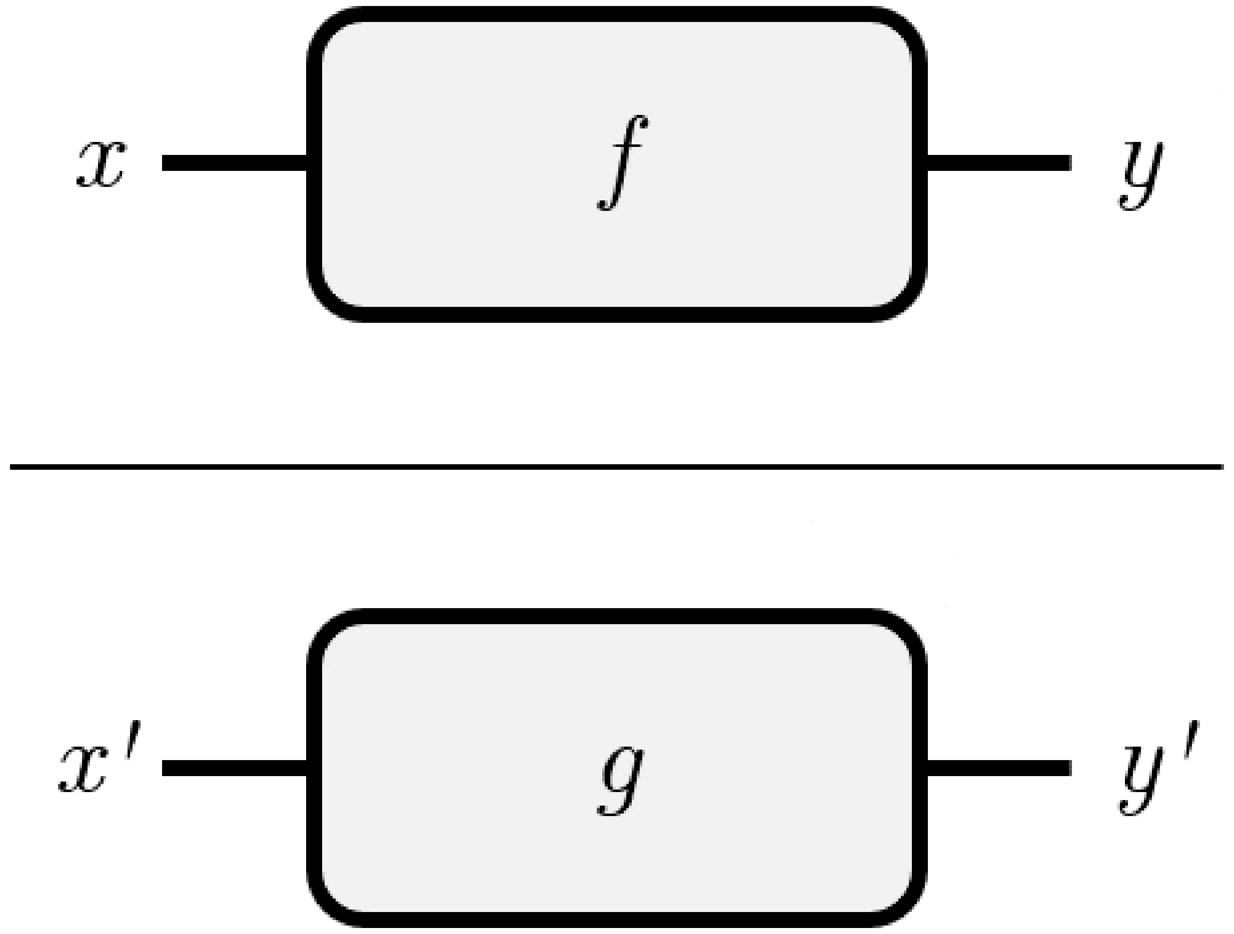
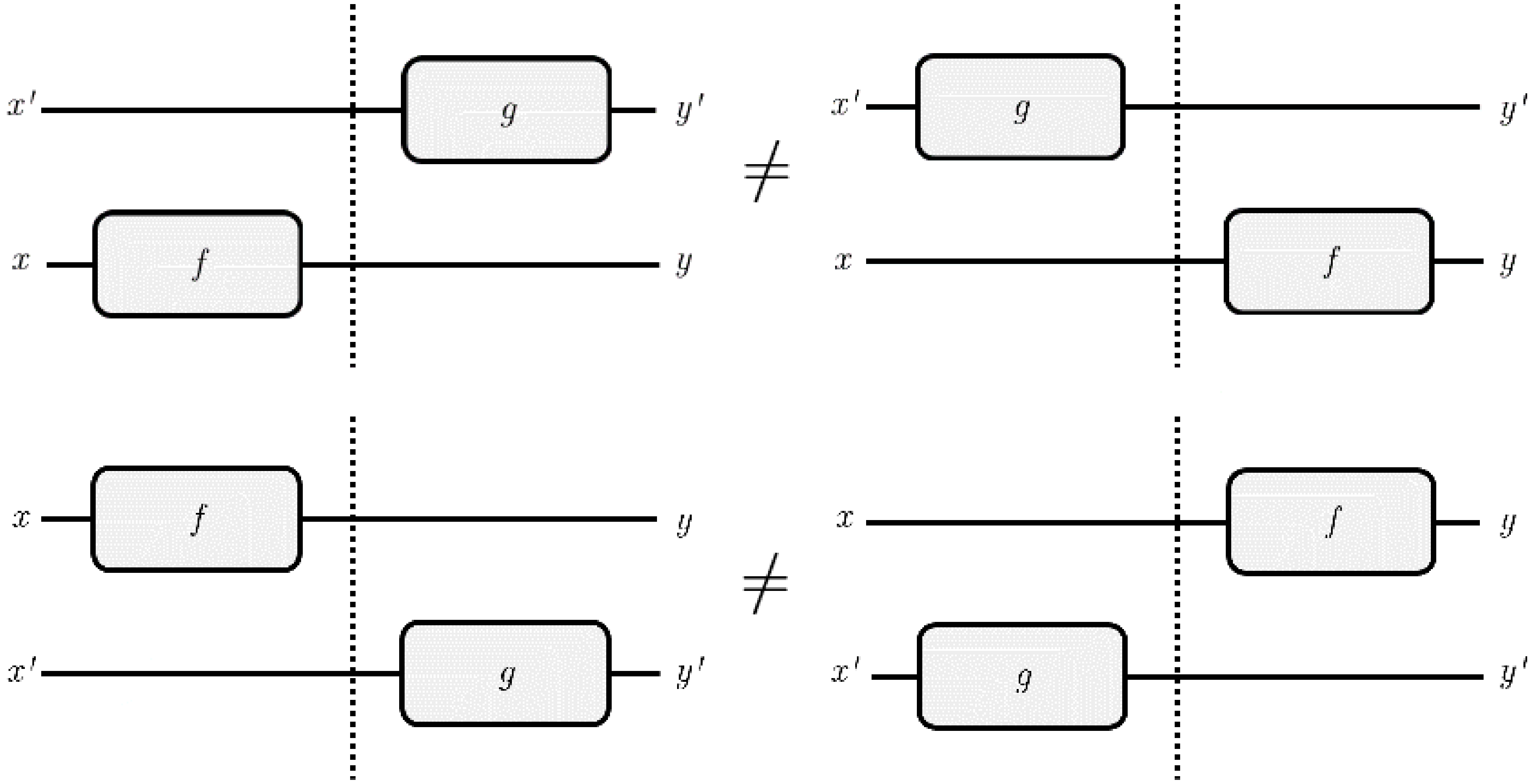
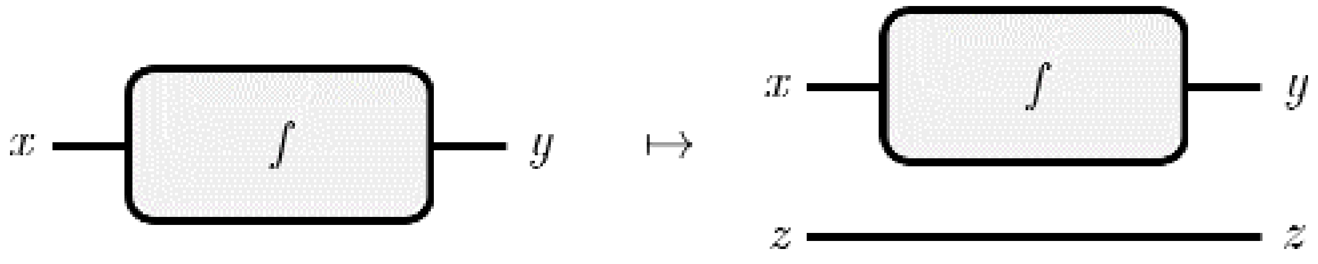




$$(g \ltimes y).(x' \rtimes f) \neq (y' \rtimes f).(g \ltimes x) \qquad (y \rtimes g).(f \ltimes x') \neq (f \ltimes y').(x \rtimes g)$$



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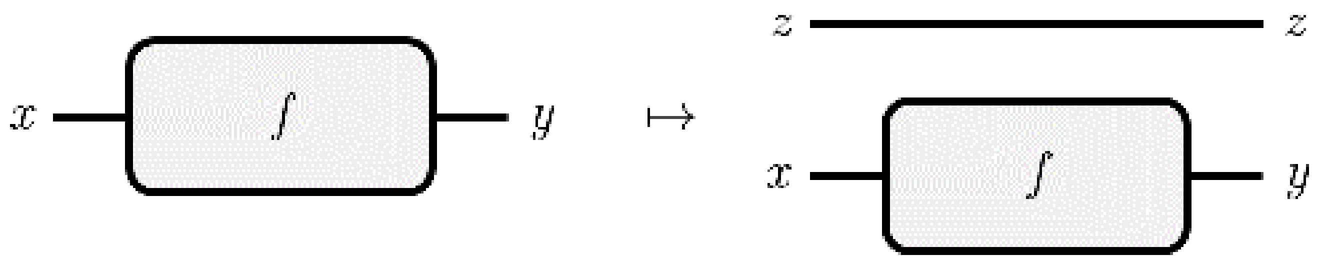

$f \mapsto z \rtimes f$

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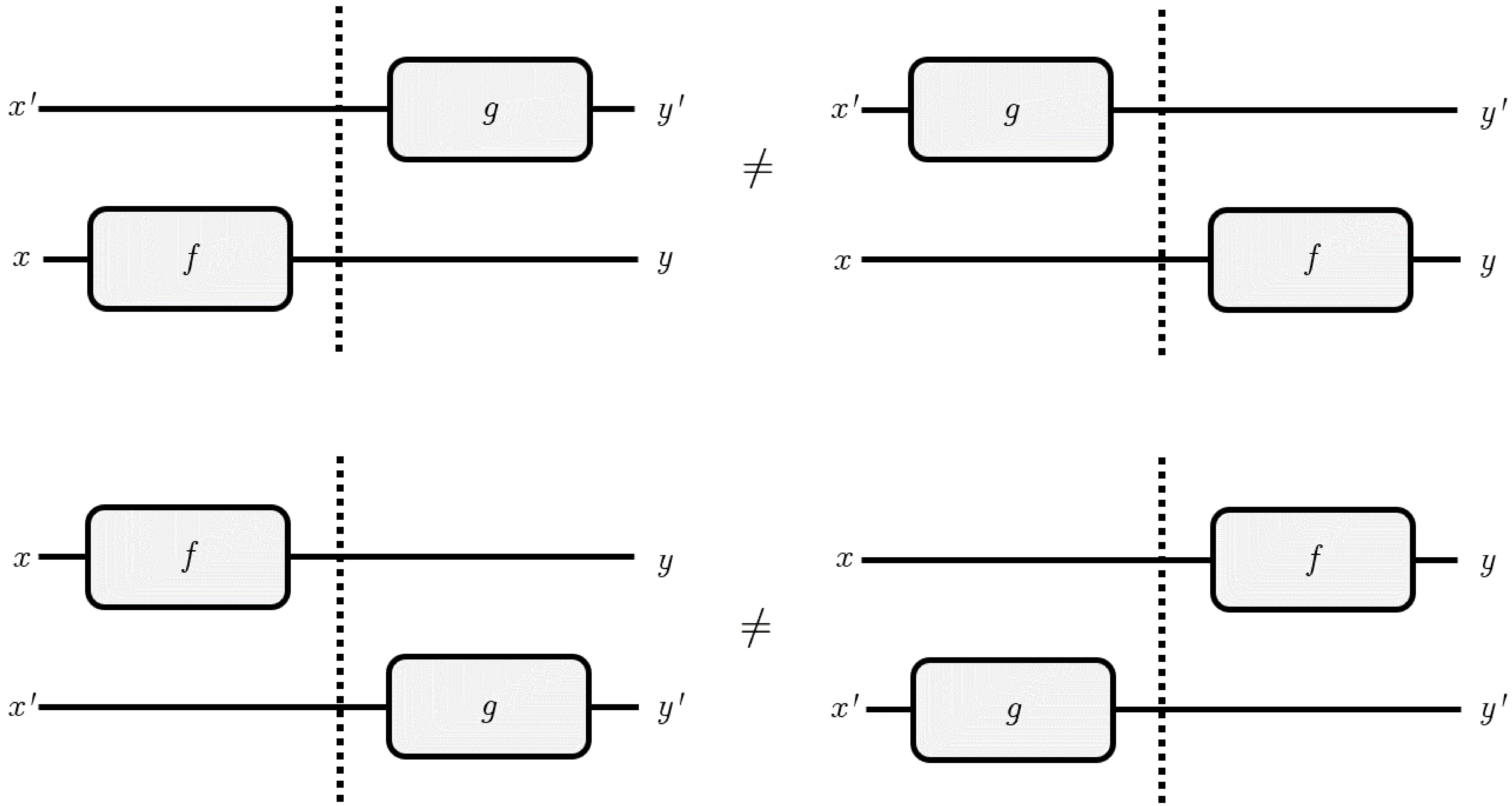

$(g \ltimes y).(x' \rtimes f) \neq (y' \rtimes f).(g \ltimes x)$ $\quad$ $(y \rtimes g).(f \ltimes x') \neq (f \ltimes y').(x \rtimes g)$

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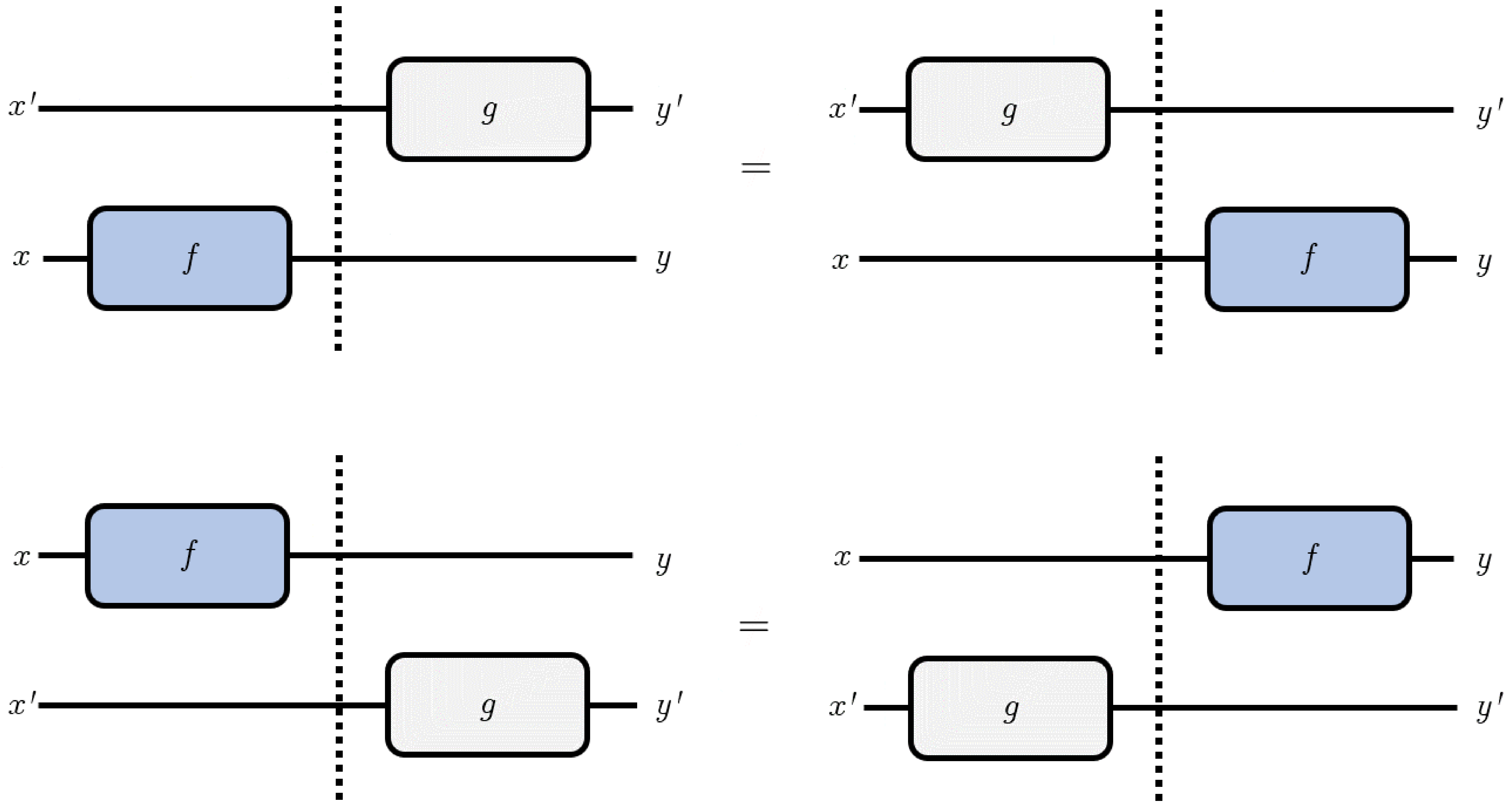

$$(g \ltimes y).(x' \rtimes f) = (y' \rtimes f).(g \ltimes x) \qquad (y \rtimes g).(f \ltimes x') = (f \ltimes y').(x \rtimes g)$$

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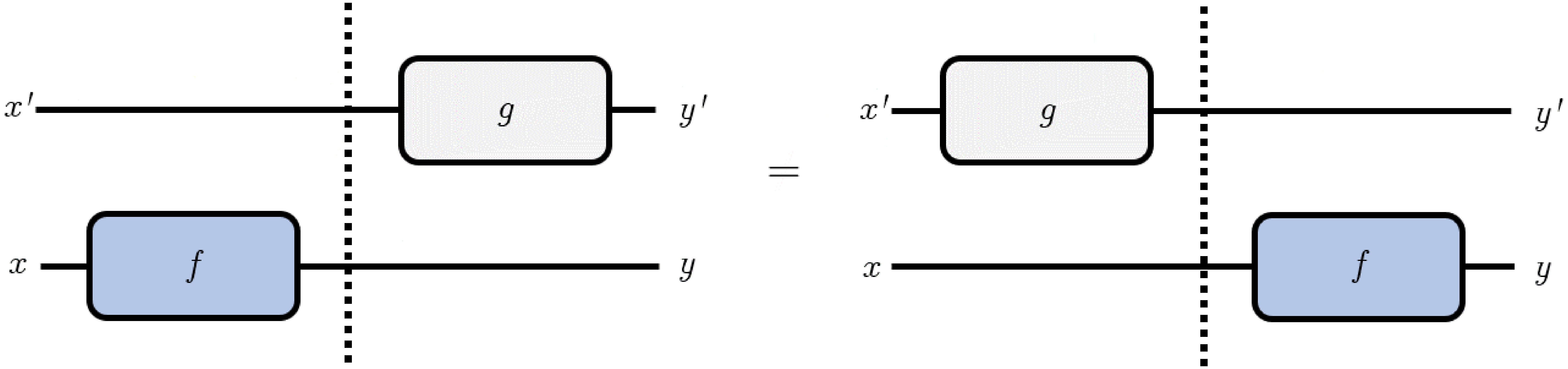
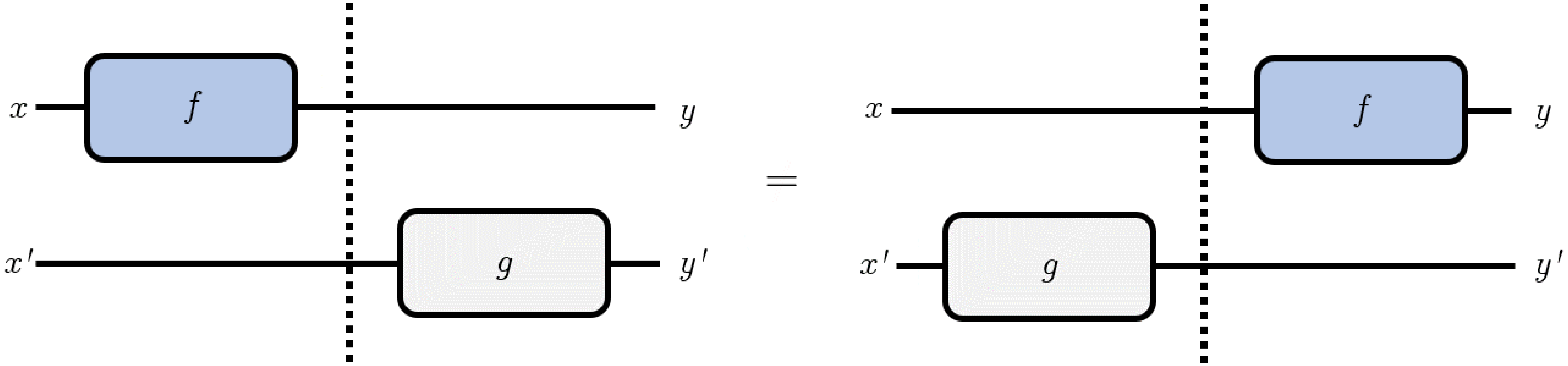


f is *central* if this holds for all g



$(g \ltimes y).(x' \rtimes f) = (y' \rtimes f).(g \ltimes x) \qquad (y \rtimes g).(f \ltimes x') = (f \ltimes y').(x \rtimes g)$

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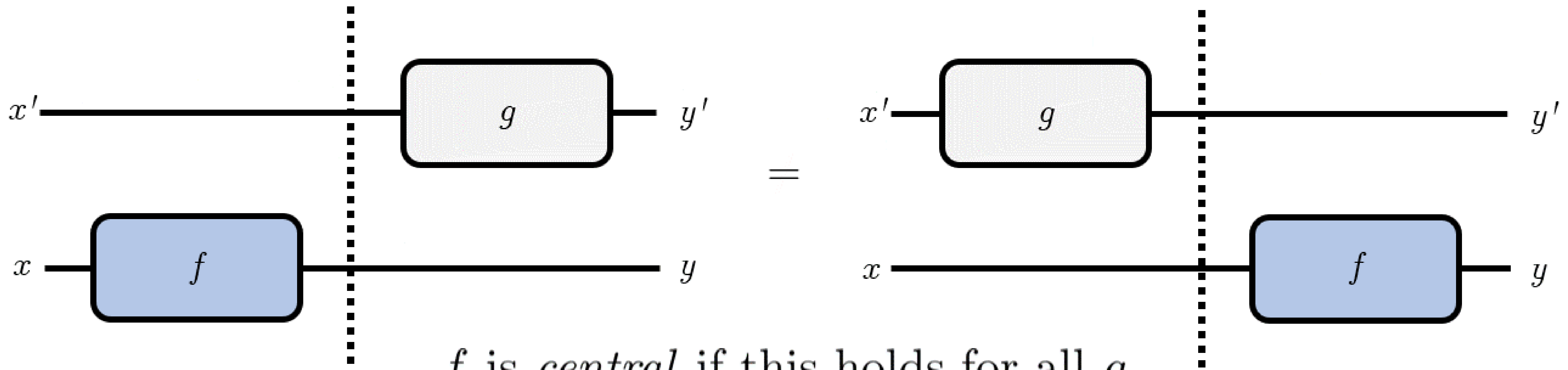
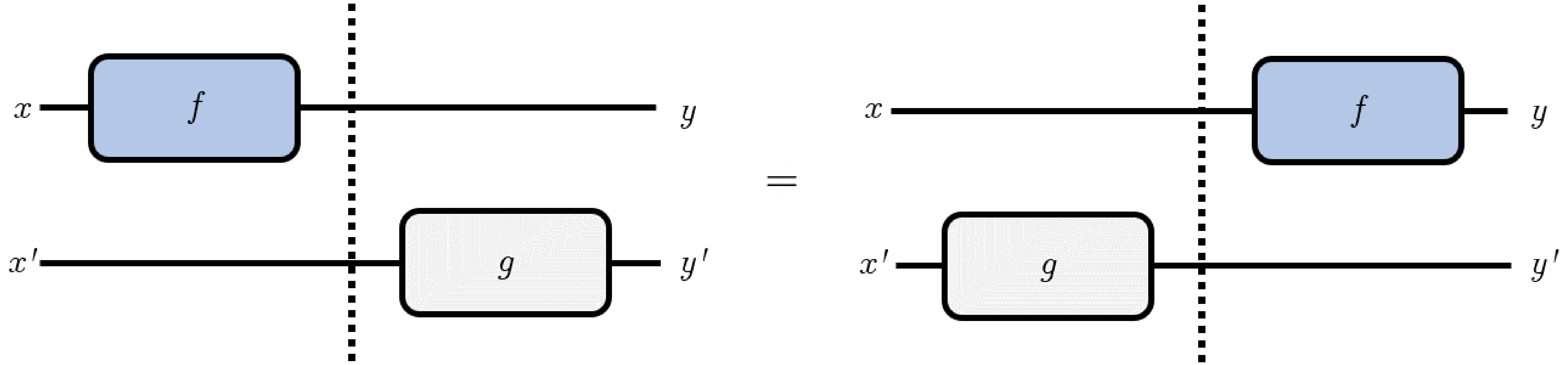


f is *central* if this holds for all g

$Z(\mathbf{C})$ is the wide subcategory of central morphisms



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Premonoidal Categories

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$\rho\colon (x \otimes e) \to x$ a central, natural isomorphism

$\lambda\colon (e \otimes x) \to x$ a central, natural isomorphism

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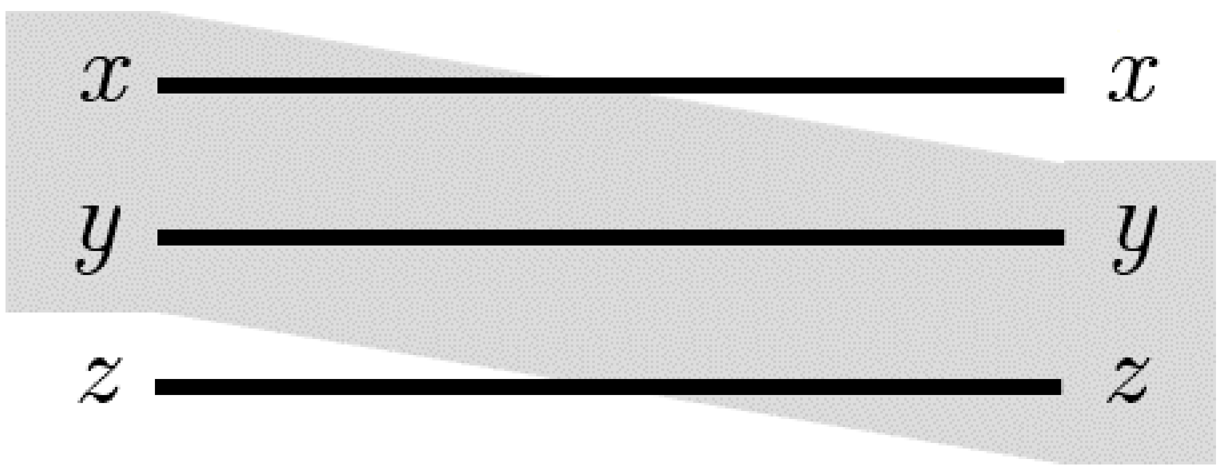

$\alpha\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ a central, natural isomorphism

Premonoidal Categories

$\rho\colon (x \otimes e) \to x$ a central, natural isomorphism

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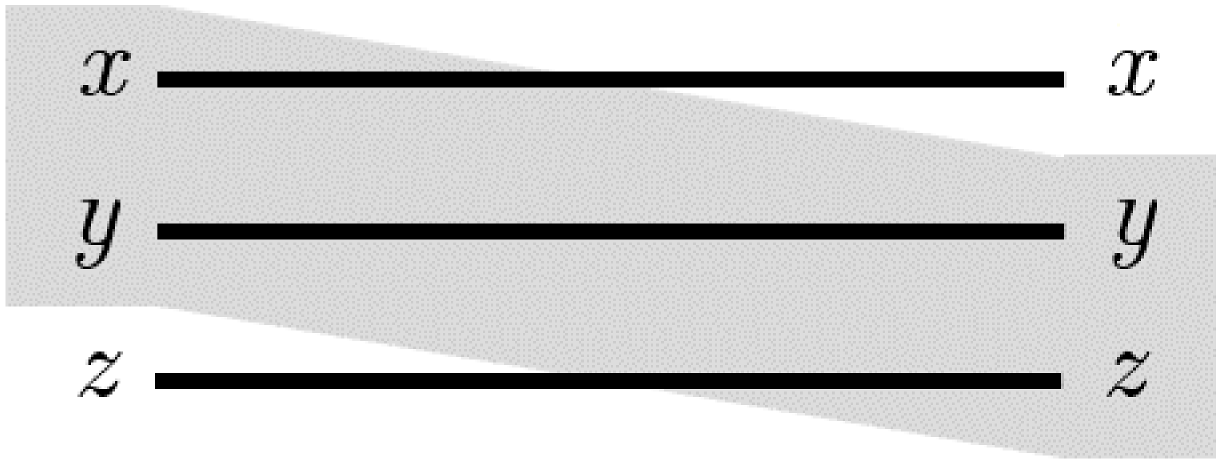

$\alpha\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ a central, natural isomorphism

satisfying the triangle and pentagon equations

(Pre)monoidal Functor

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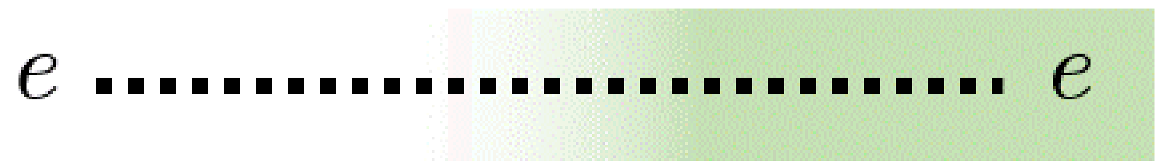
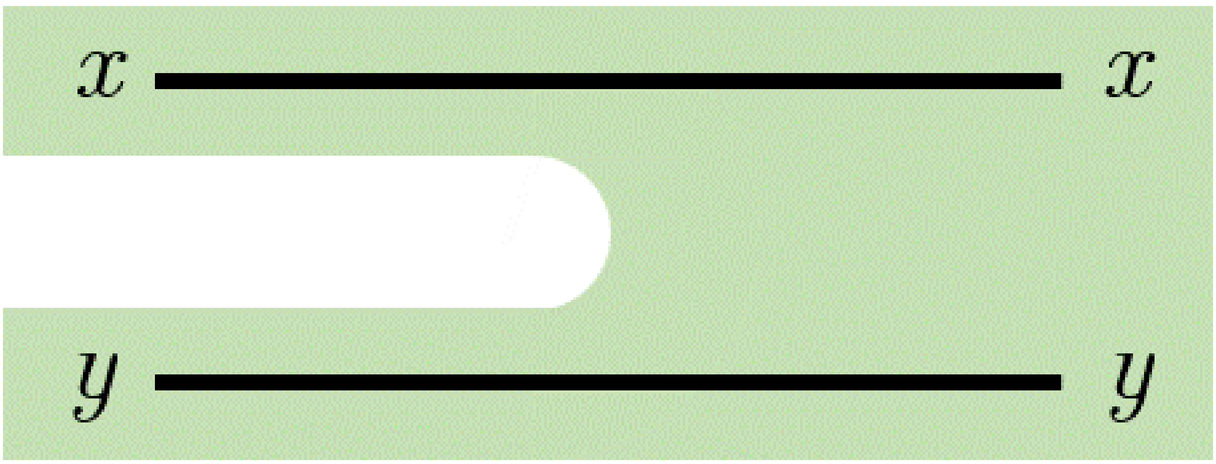
$\eta\colon e_{\mathbf{D}}$



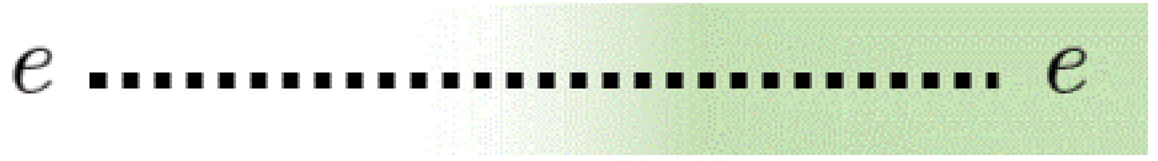
$F(e_{\mathbf{C}})$

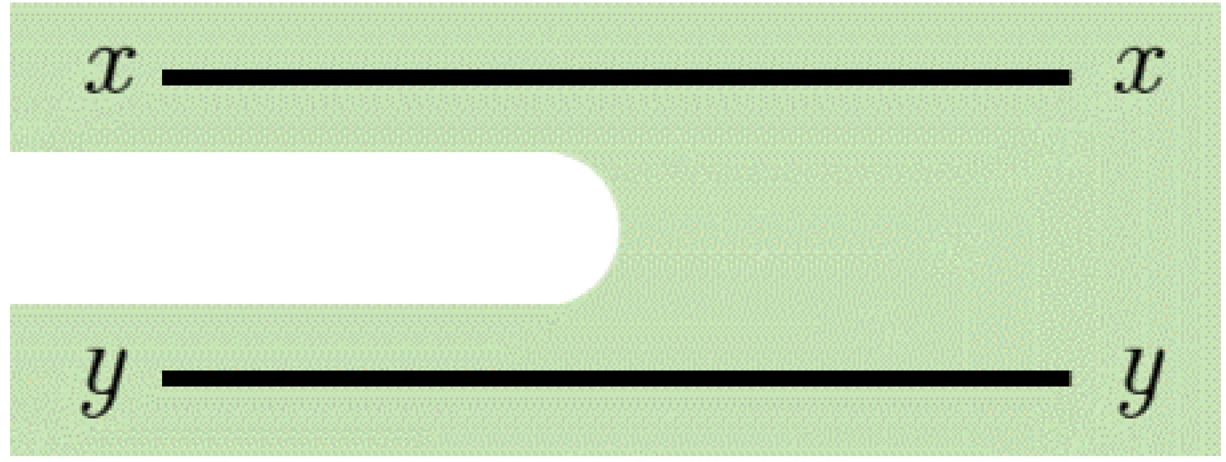
$\mu\colon F(x) \otimes_{\mathbf{D}} F(y)$

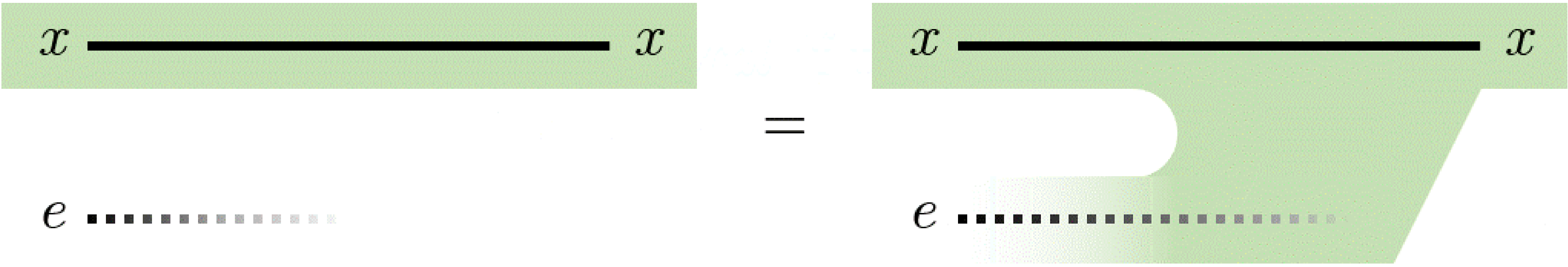


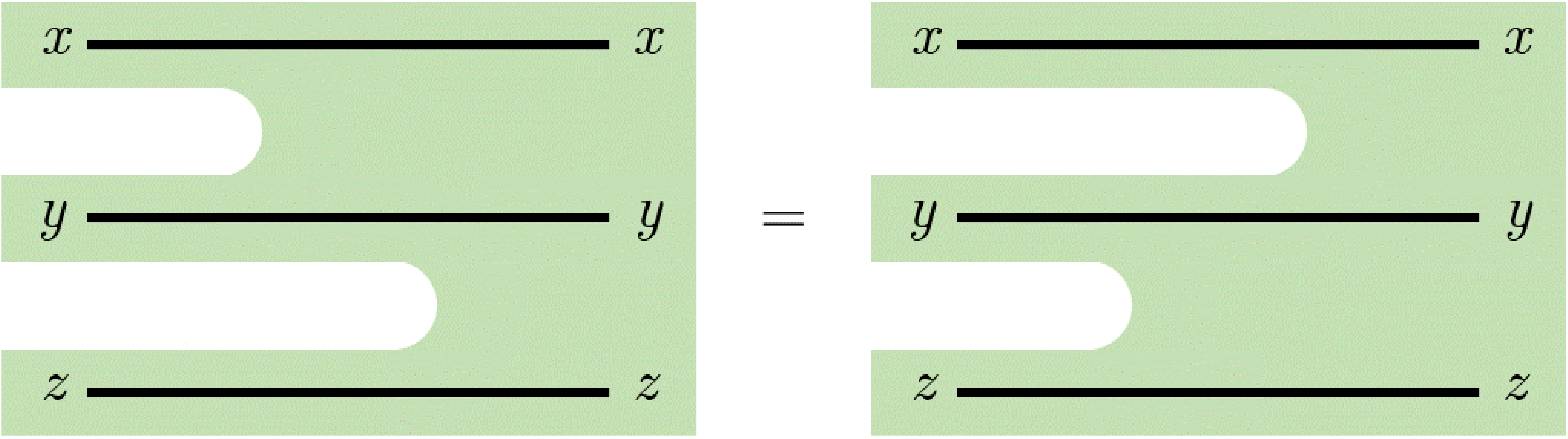
$F(x \otimes_{\mathbf{C}} y)$

(Pre)monoidal Functor

$\eta\colon e_{\mathbf{D}}$  $F(e_{\mathbf{C}})$

$\mu\colon F(x) \otimes_{\mathbf{D}} F(y)$  $F(x \otimes_{\mathbf{C}} y)$

$e_{\mathbf{D}} \otimes_{\mathbf{D}} F(x)$  $F(x)$ and symmetric

$(F(x) \otimes_{\mathbf{D}} F(y)) \otimes_{\mathbf{D}} F(z)$  $F(x \otimes_{\mathbf{C}} (y \otimes_{\mathbf{C}} z))$

Freyd Categories

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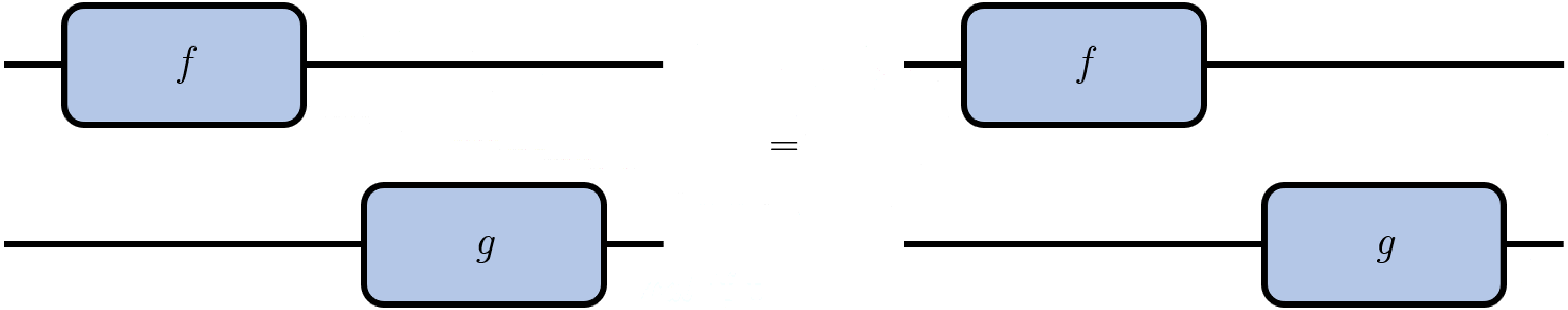

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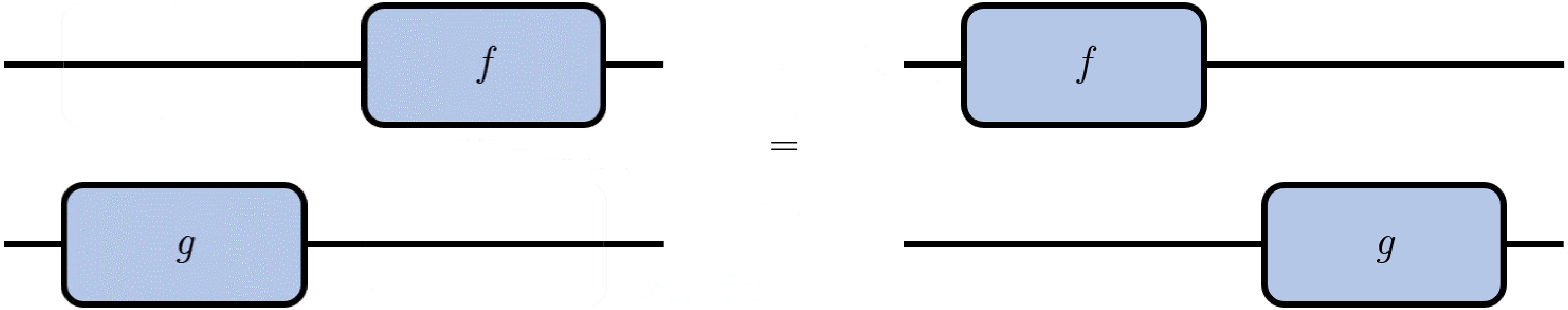
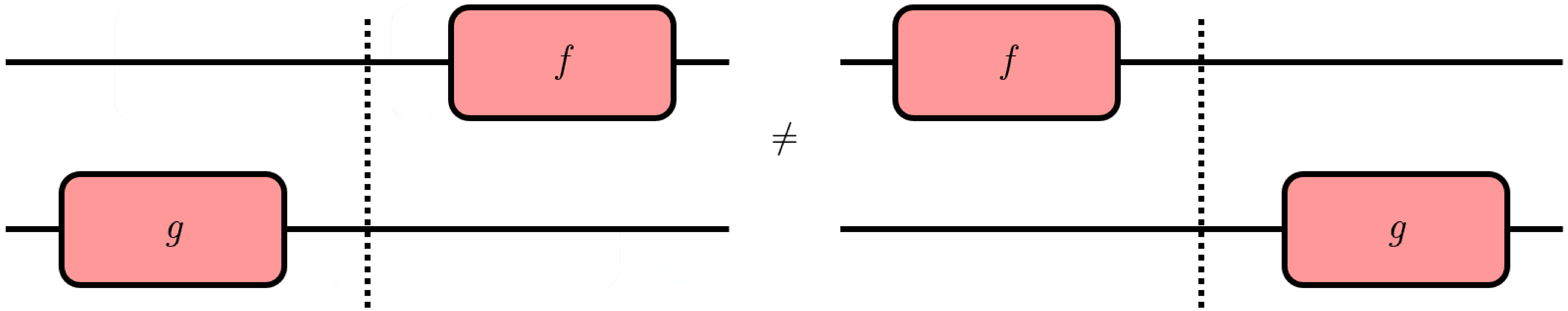
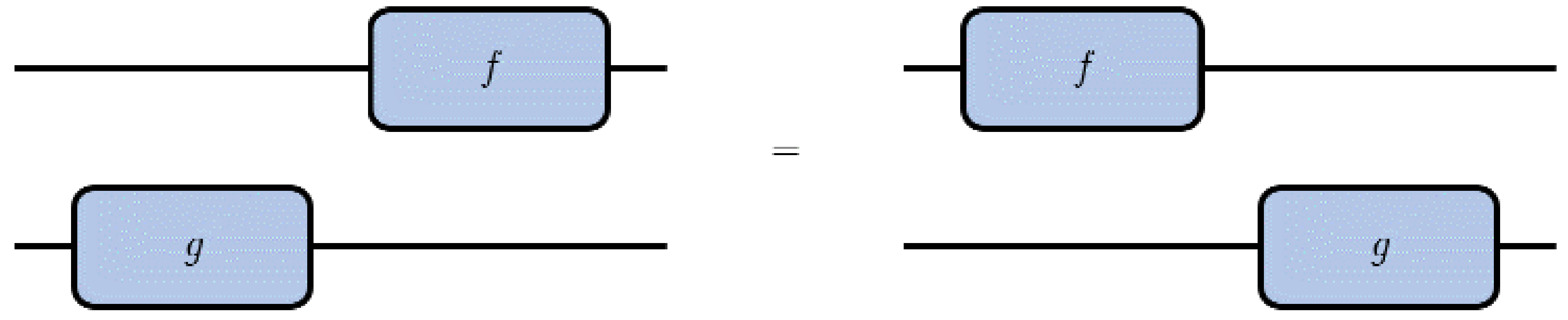


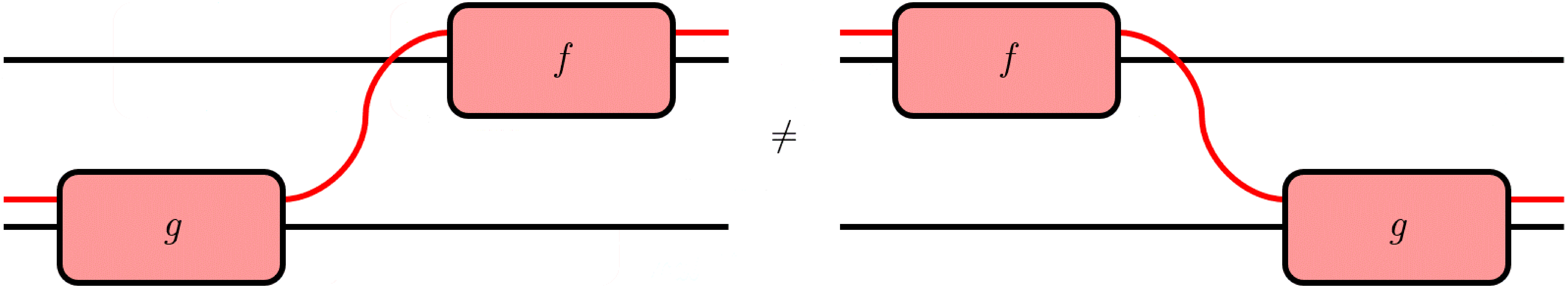
Freyd Categories

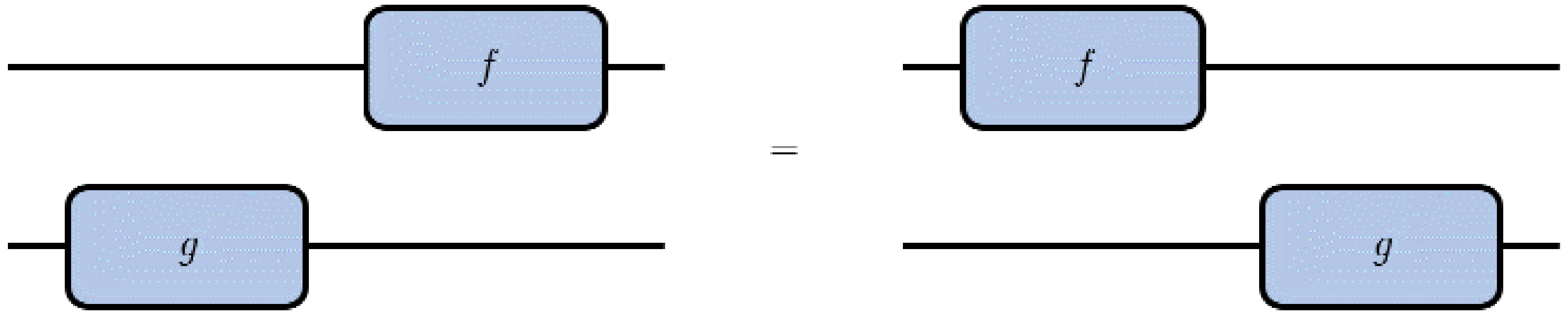




monoidal $\mathbf{M}$

Freyd Categories

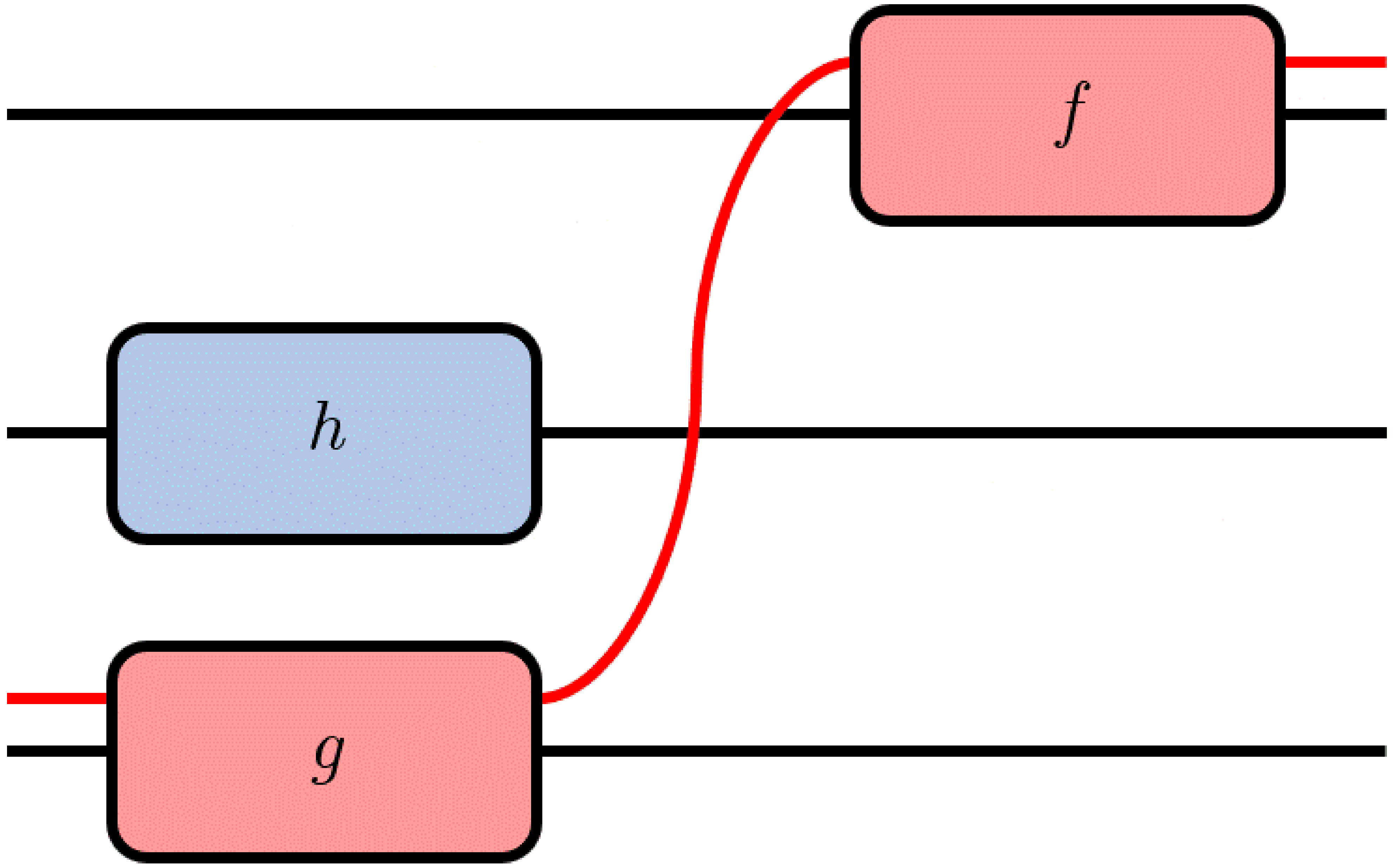


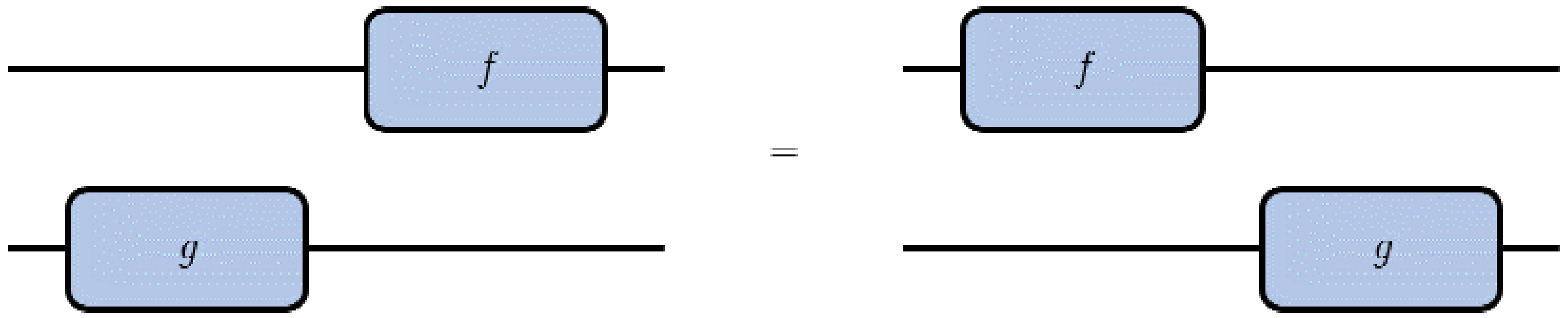


monoidal $\mathbf{M}$

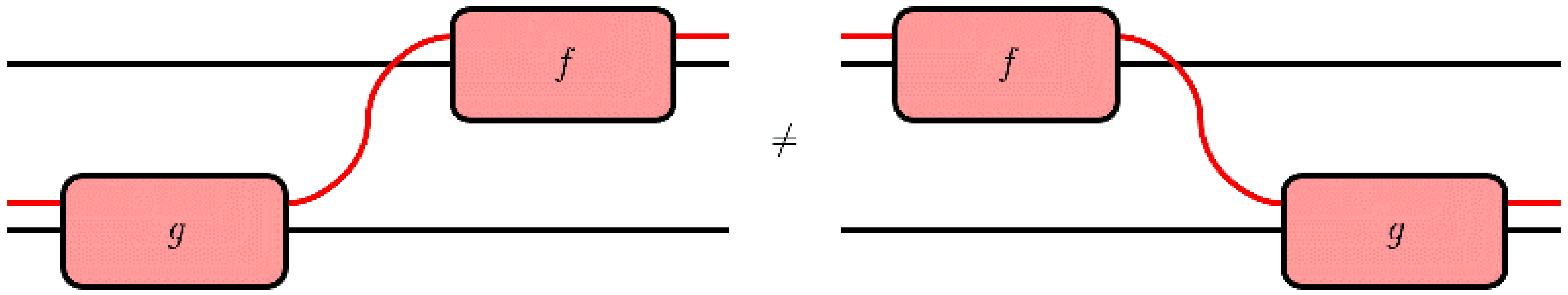
Freyd Categories

an identity-on-objects strict premonoidal functor $J \colon \mathbf{M} \to \mathbf{C}$ whose image lies in $Z(\mathbf{C})$





monoidal **M**



premonoidal **C**

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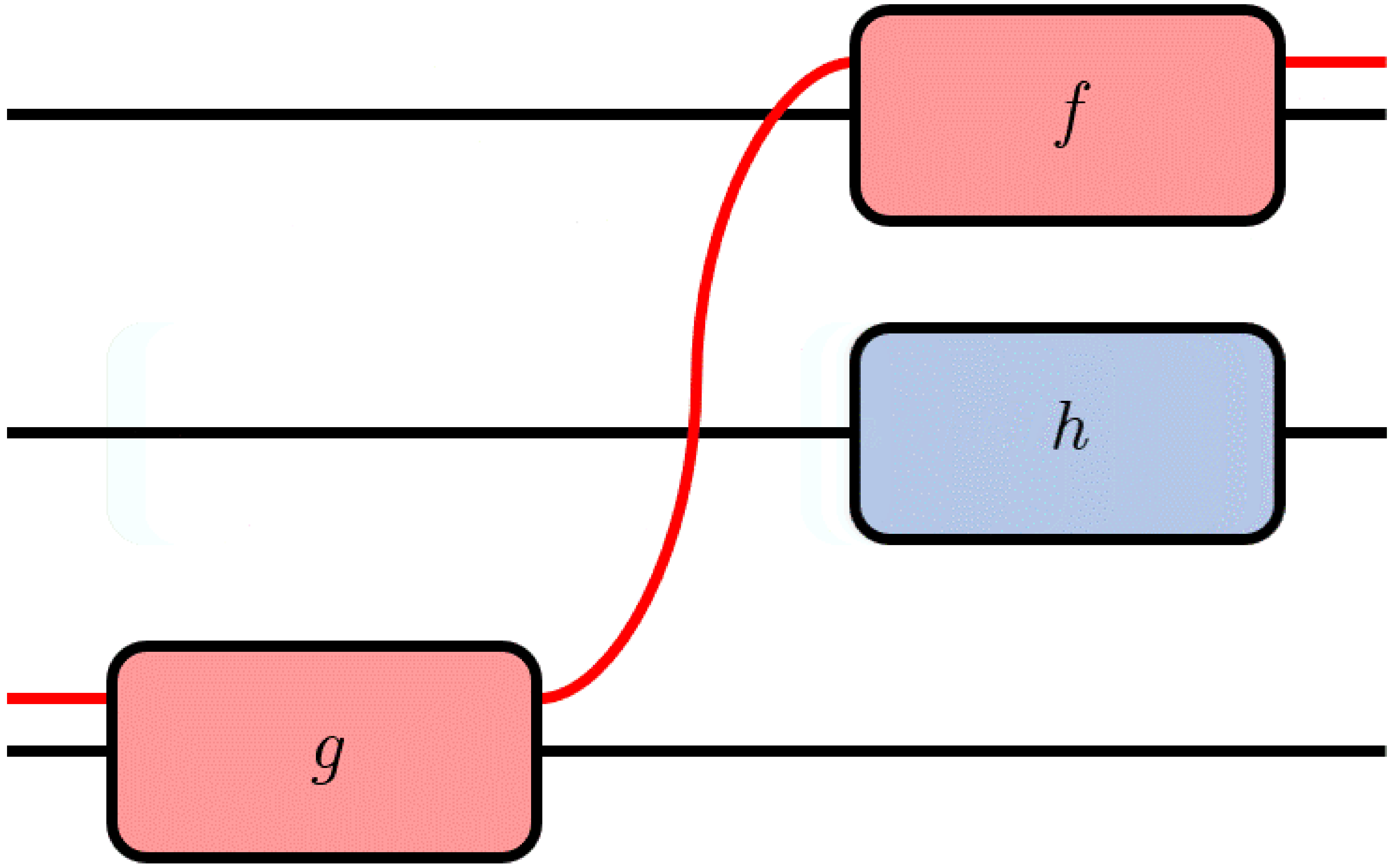
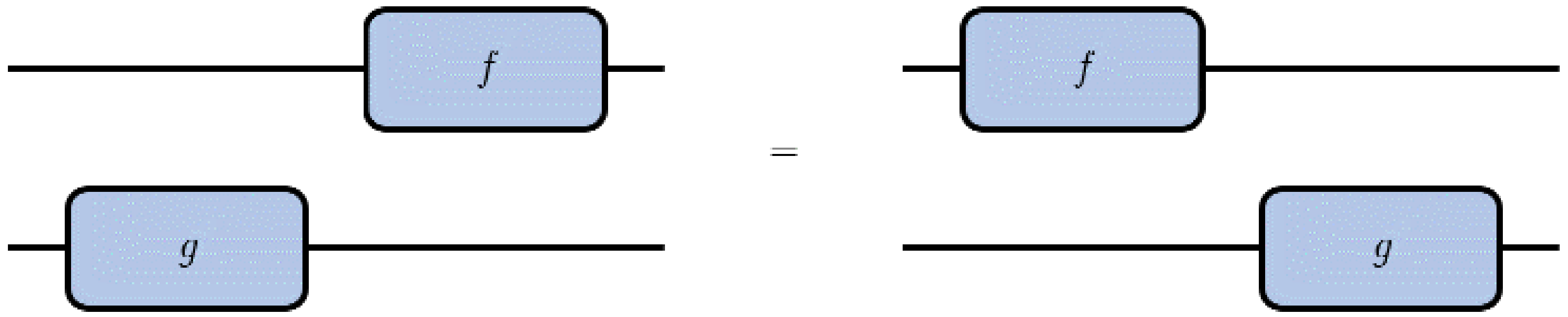
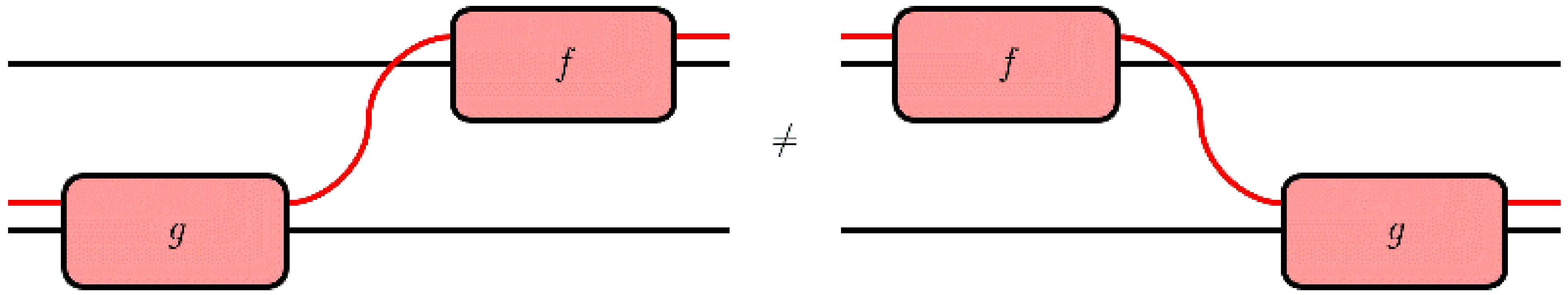




monoidal $\mathbf{M}$



premonoidal $\mathbf{C}$

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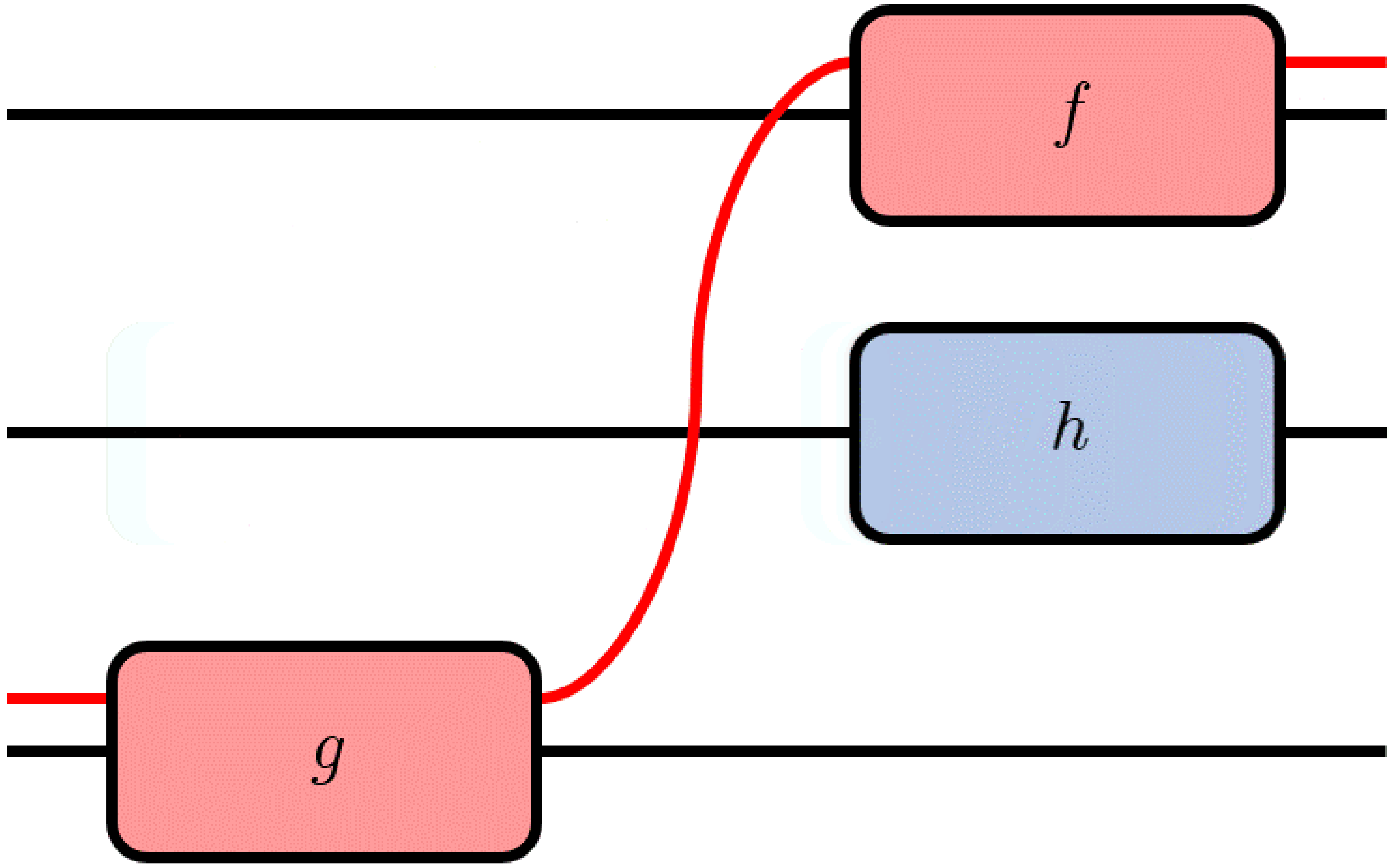
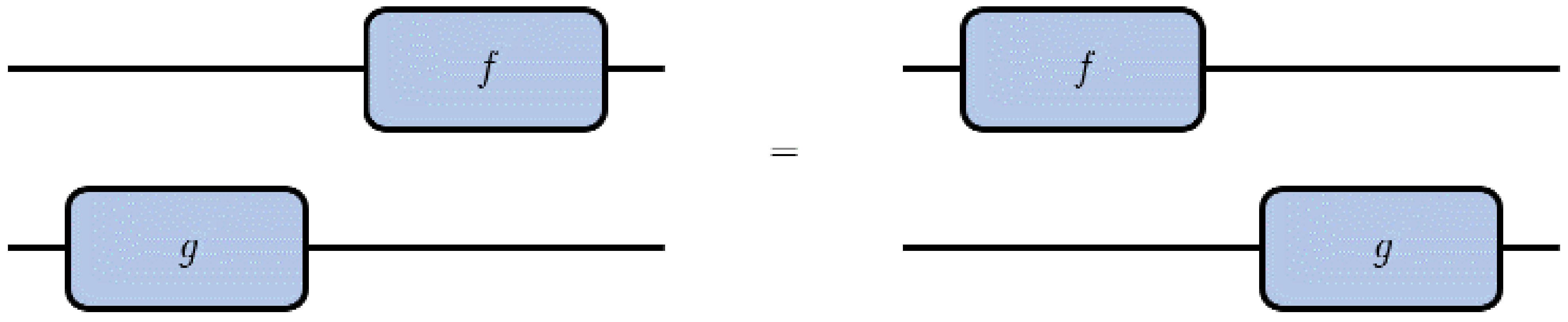
an identity-on-objects strict premonoidal functor $J\colon \mathbf{M} \to \mathbf{C}$ whose image lies in $Z(\mathbf{C})$

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{F_0} & \mathbf{M}' \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J'} \\ \mathbf{C} & \xrightarrow[F_1]{} & \mathbf{C}' \end{array}$$

F_0 strong monoidal
F_1 strong premonoidal





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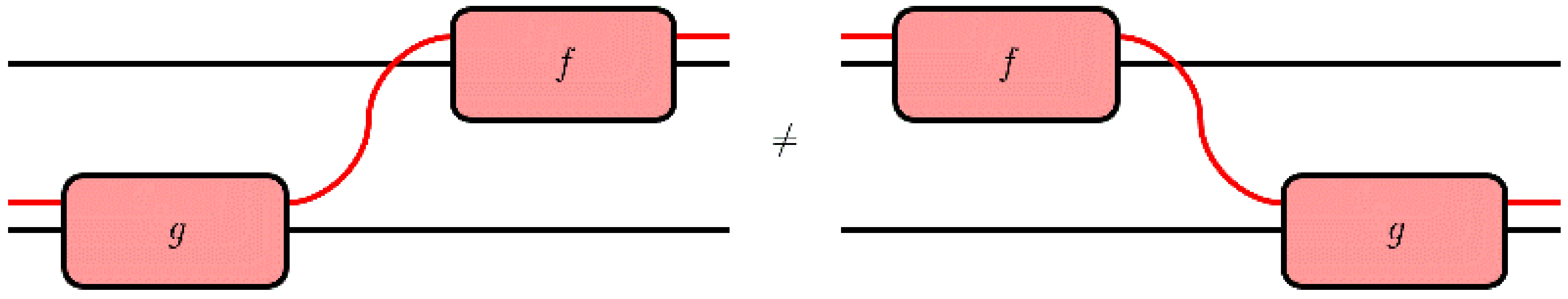
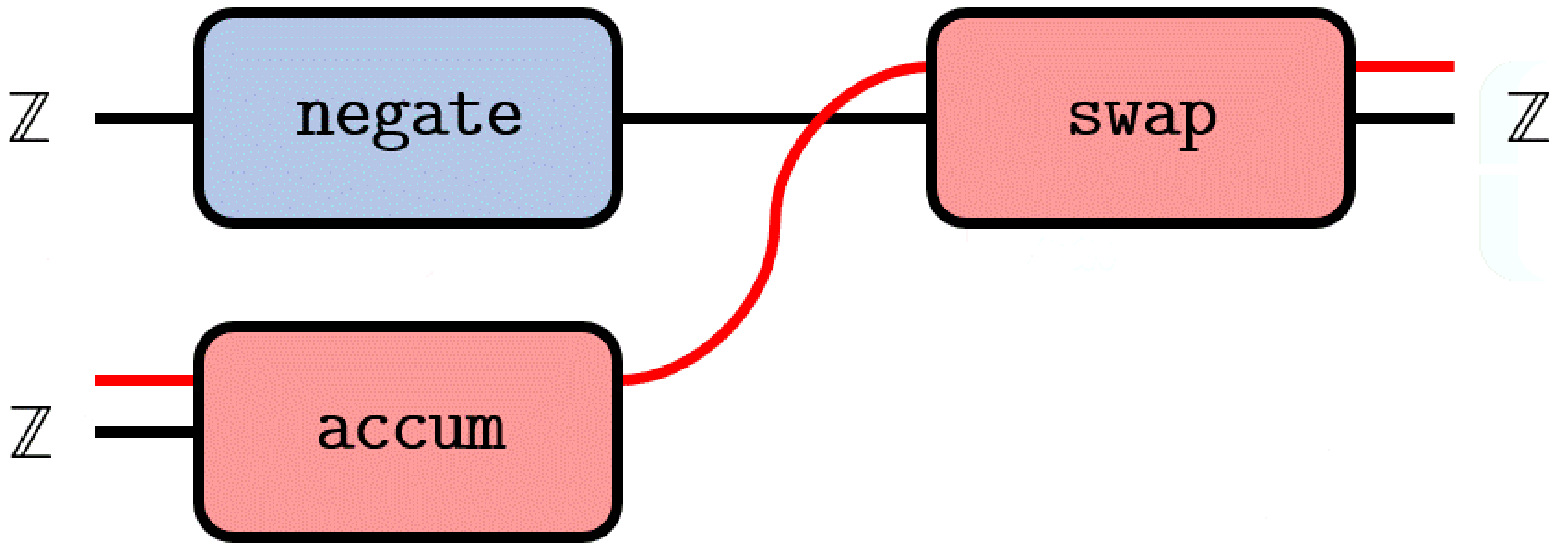


premonoidal $\mathbf{C}$

Freyd Category Example

Freyd Category Example

$\mathbf{M} = (\mathbf{Set}, \times, 1)$ $\mathbf{C}(a,b) := \mathbf{Set}(\mathbb{Z} \times a, \mathbb{Z} \times b)$ $f \ltimes c := f \times \mathrm{id}_c$ $J(f) := \mathrm{id}_{\mathbb{Z}} \times f$




```
negate(x) = -x

accum(s, y) = (s + y, ())

swap(s, z) = (z, s)
```

Duoidal Categories

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$(\mathbf{V}, *, J)$, parallel $\qquad$ $(\mathbf{V}, \circ, I)$, sequential

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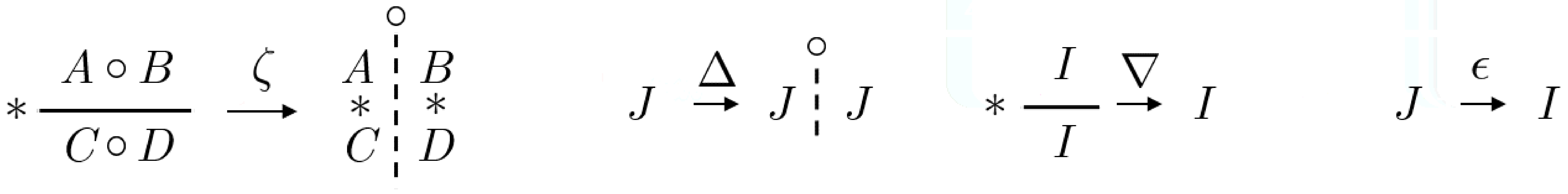
$$\zeta_{A,B,C,D} \colon (A \circ B) * (C \circ D) \to (A * C) \circ (B * D)$$

$$\Delta \colon J \to J \circ J \quad \nabla \colon I * I \to I \quad \epsilon \colon J \to I$$



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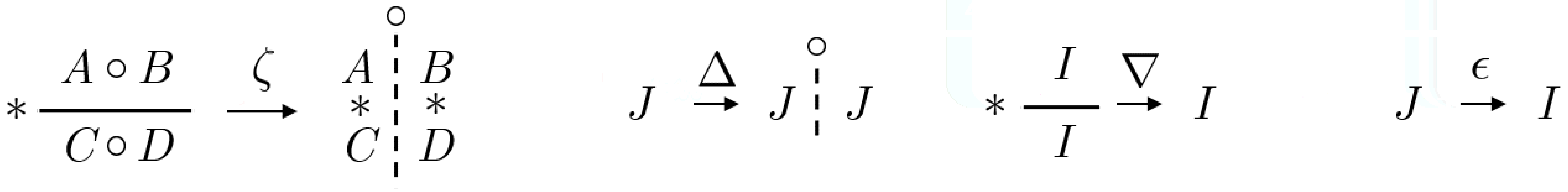

(I, ∇, ϵ) is a monoid in $(\mathbf{V}, *, J)$ $\qquad$ (J, Δ, ϵ) is a comonoid in $(\mathbf{V}, \circ, I)$

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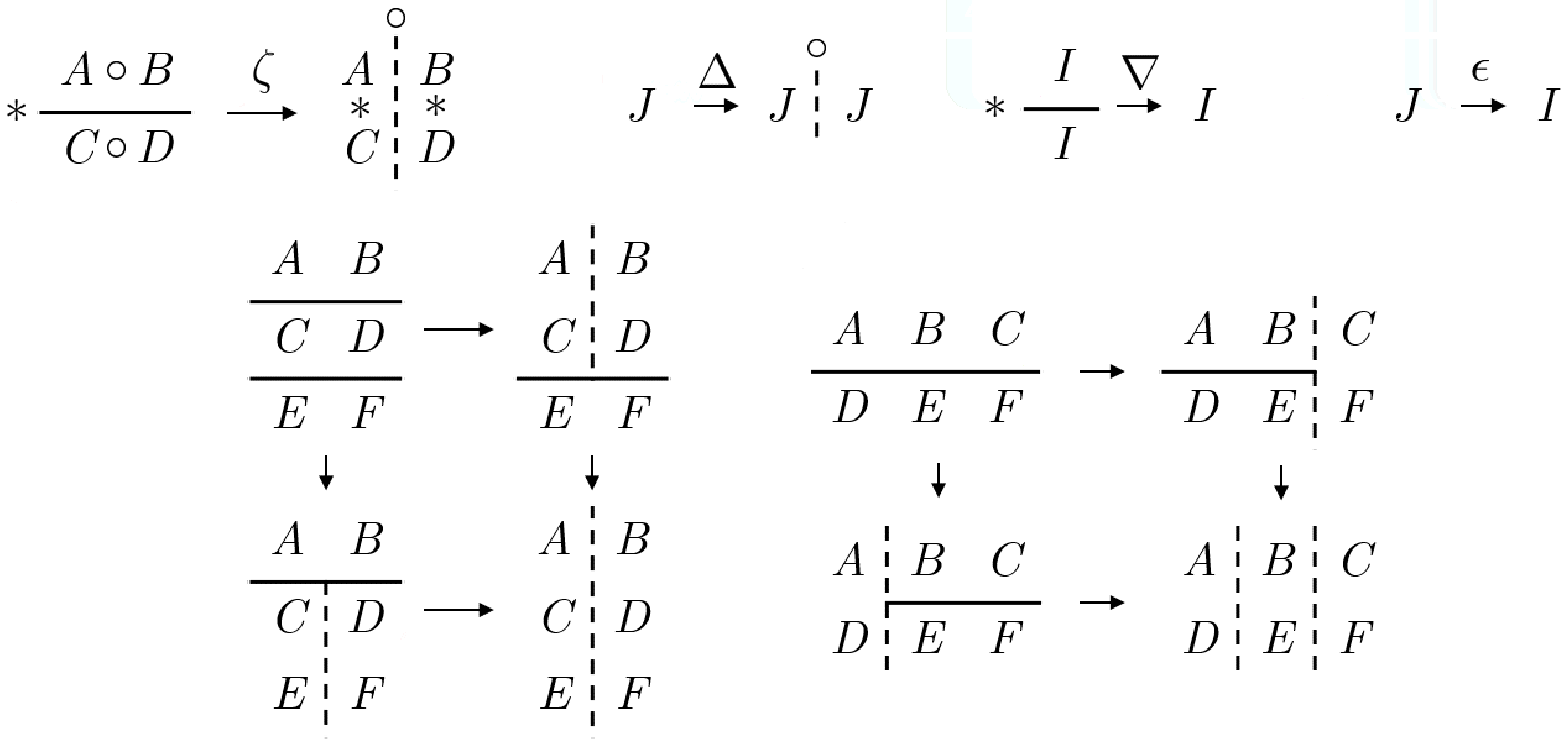

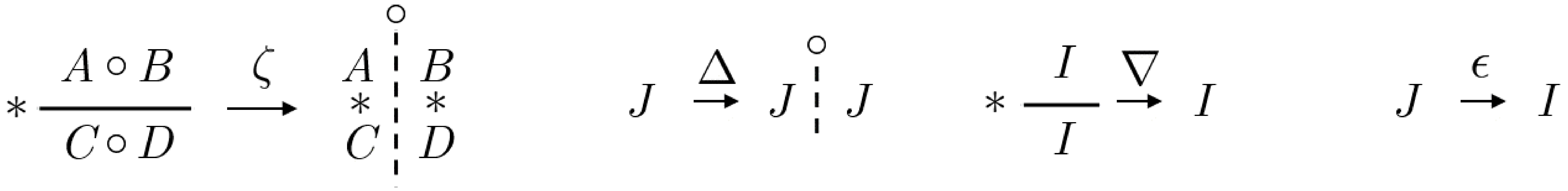

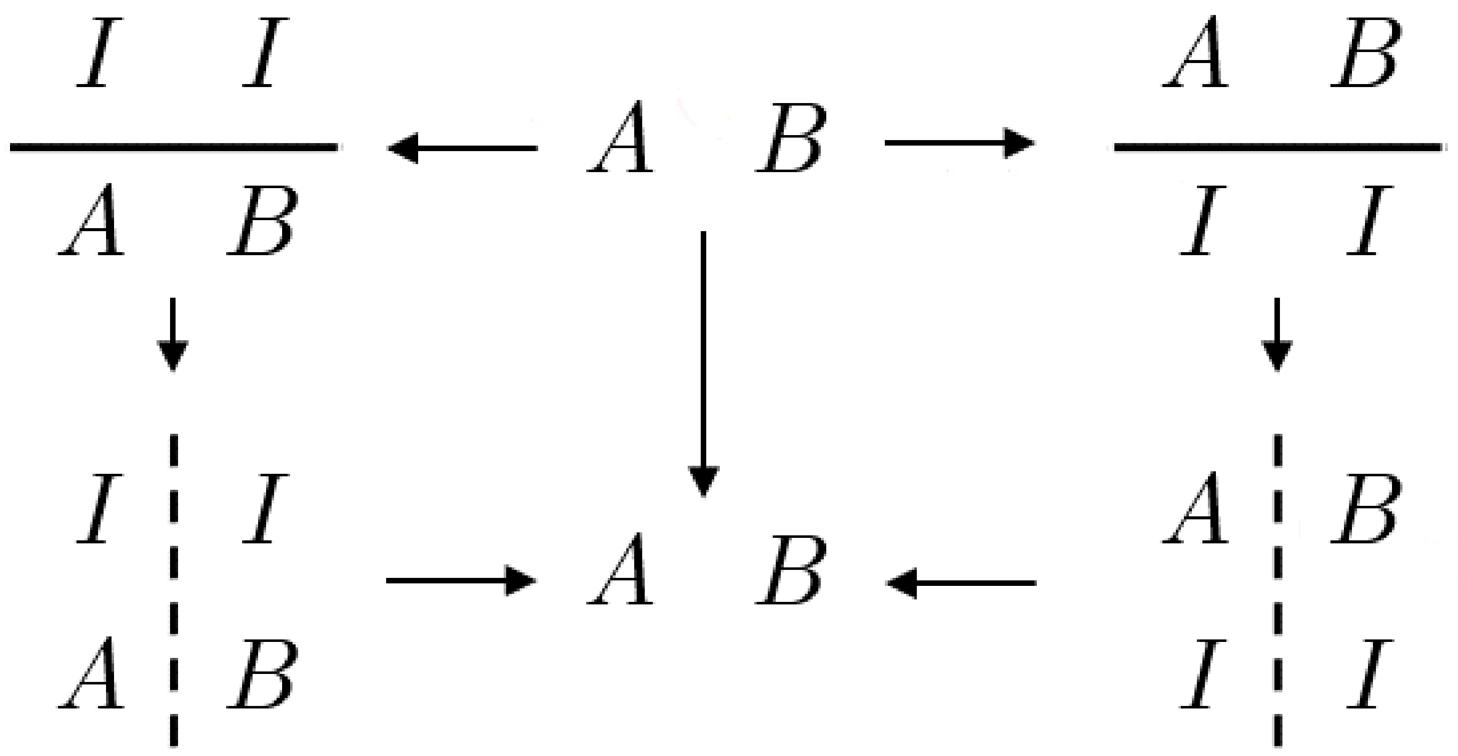

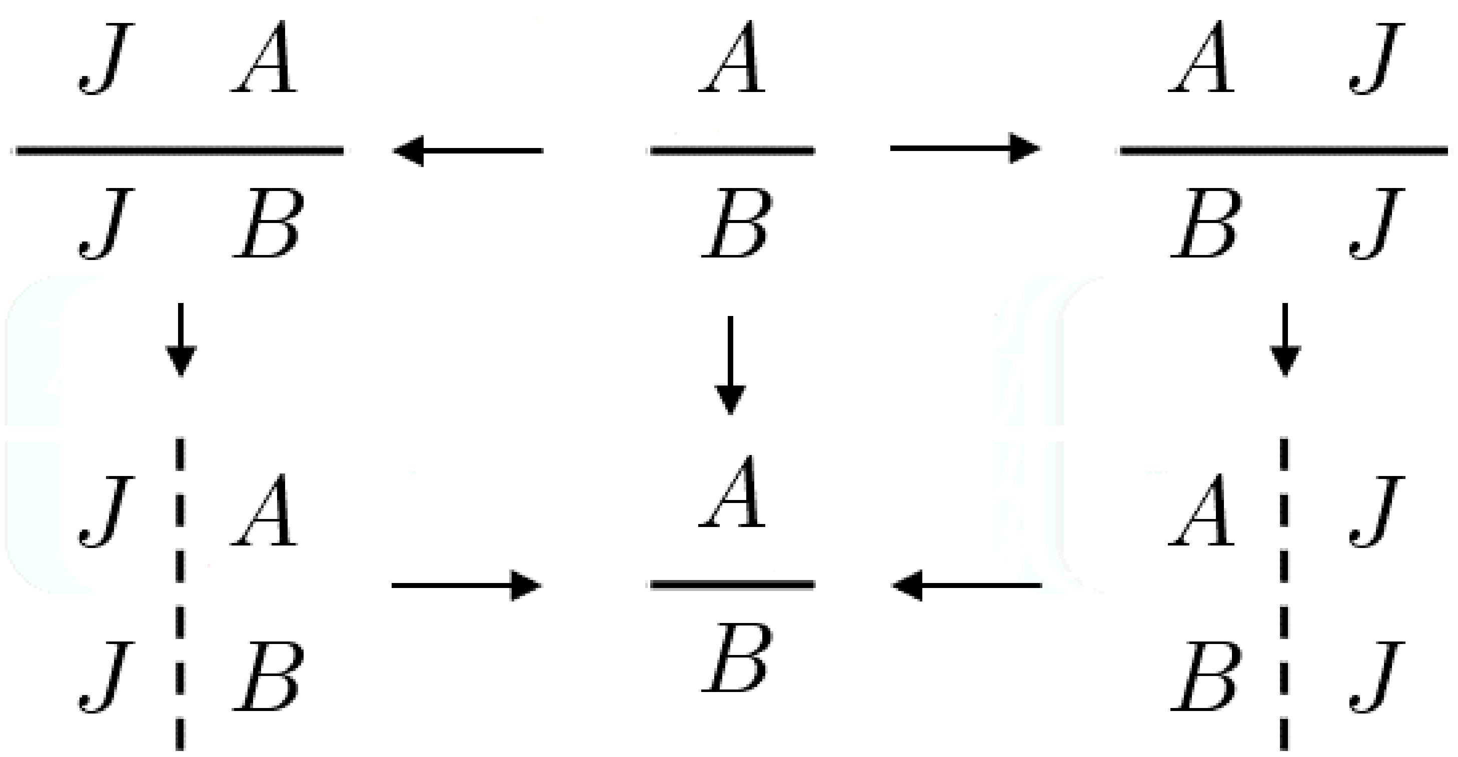

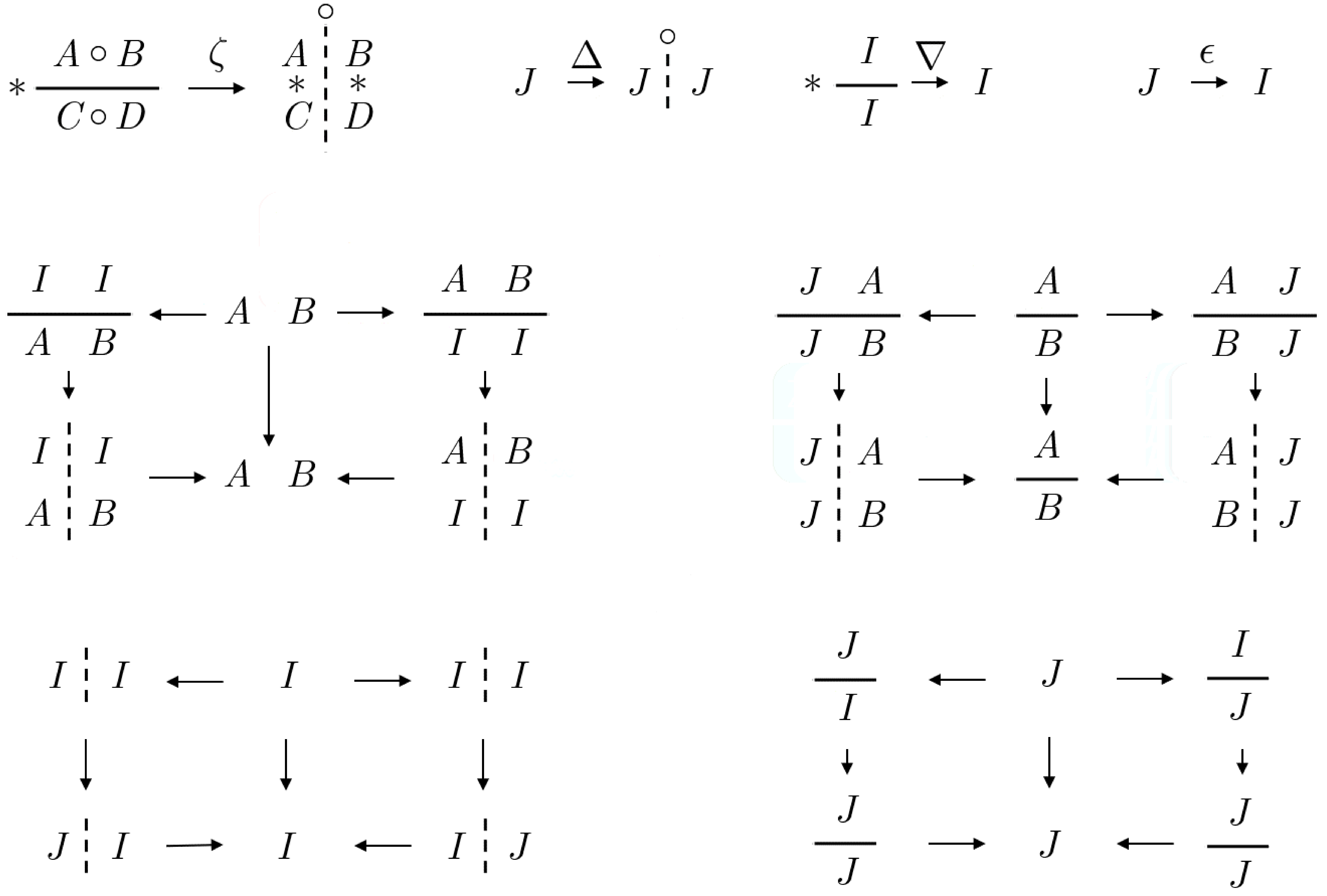

Duoidal Categories Examples

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Any braided monoidal category is duoidal with ζ being the middle-four interchange $x \otimes y \otimes z \otimes w \to x \otimes z \otimes y \otimes w$.

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For monoidal $(\mathbf{V}, \otimes, I)$ with products, $(\mathbf{V}, \otimes, I, \times, 1)$ is duoidal with $\zeta = \langle \pi_1 \otimes \pi_1, \pi_2 \otimes \pi_2 \rangle$. Similarly, $(\mathbf{V}, +, 0, \otimes, I)$ is duoidal.

Category of Labelled Sets

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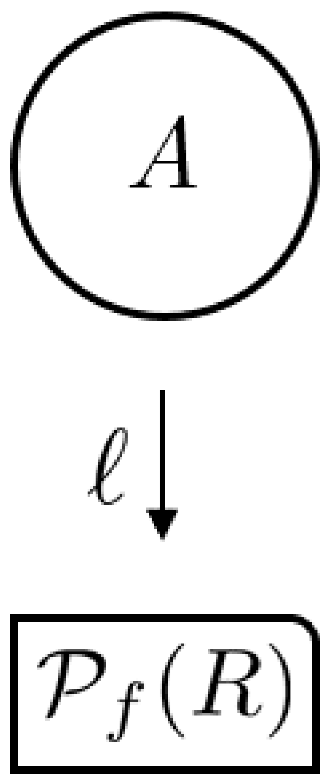

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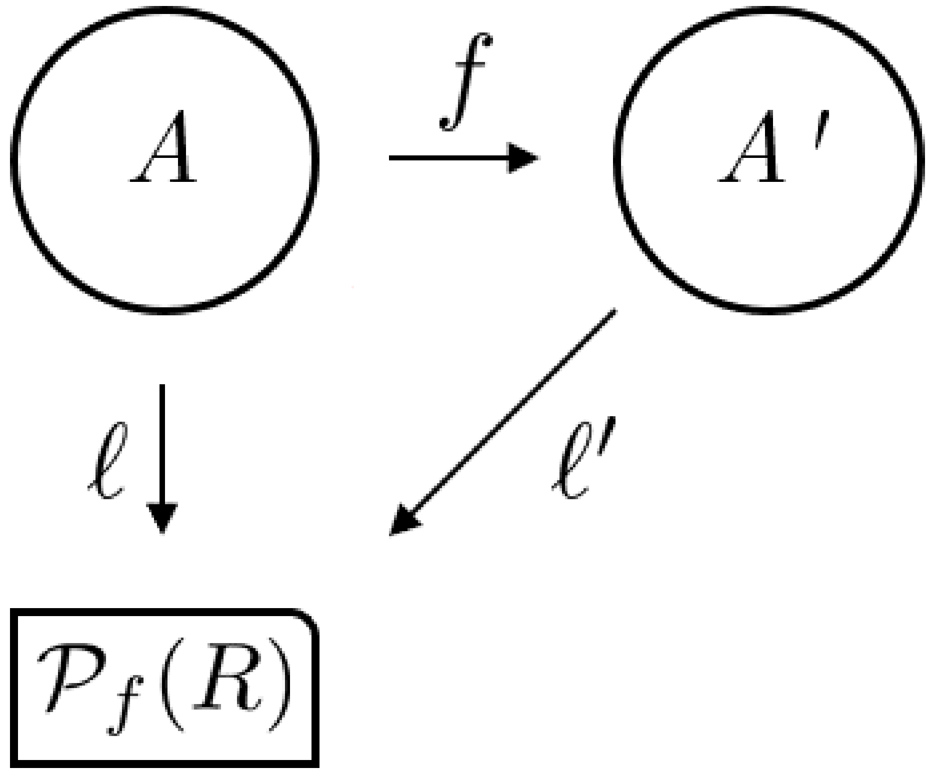

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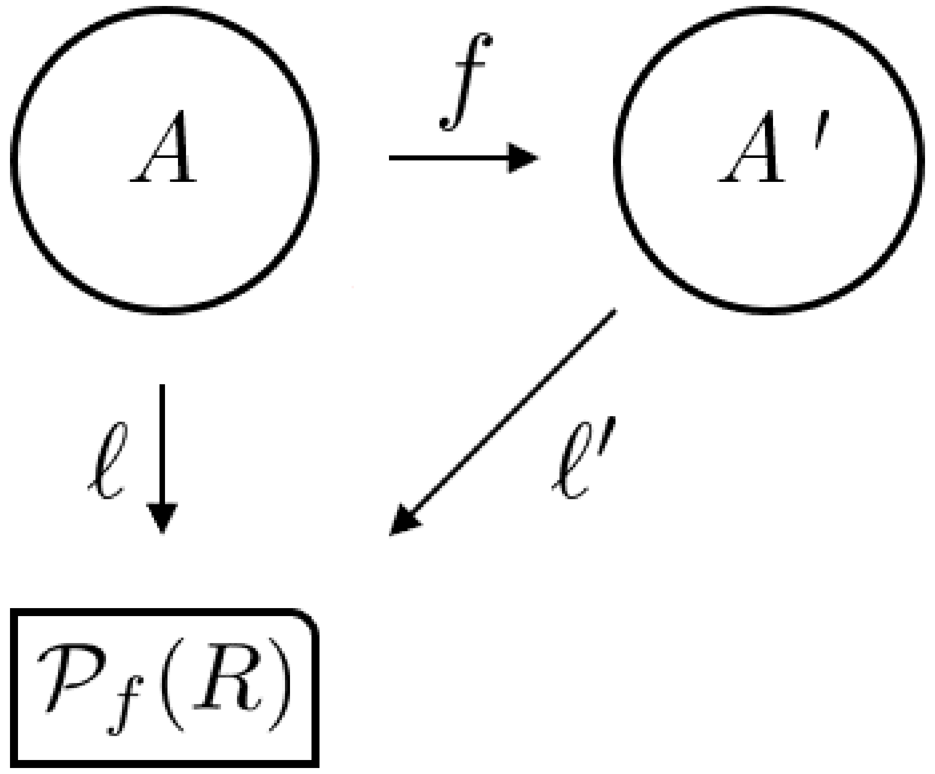

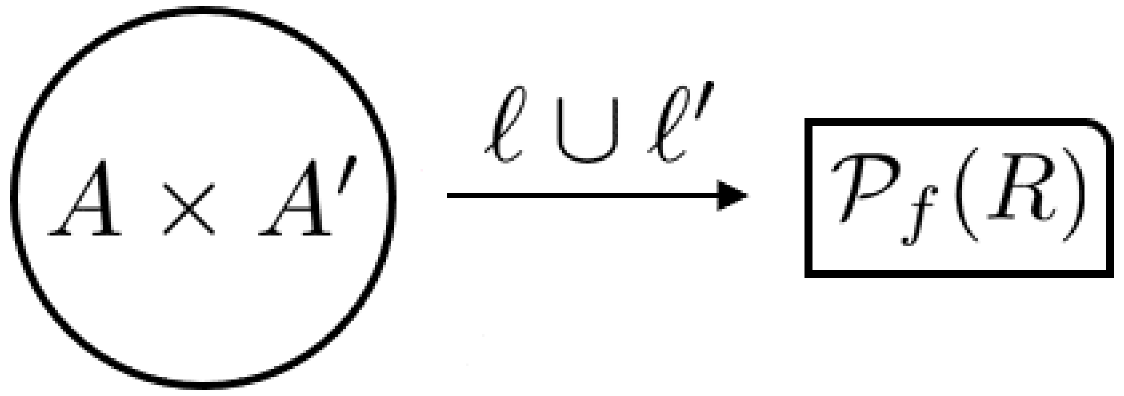

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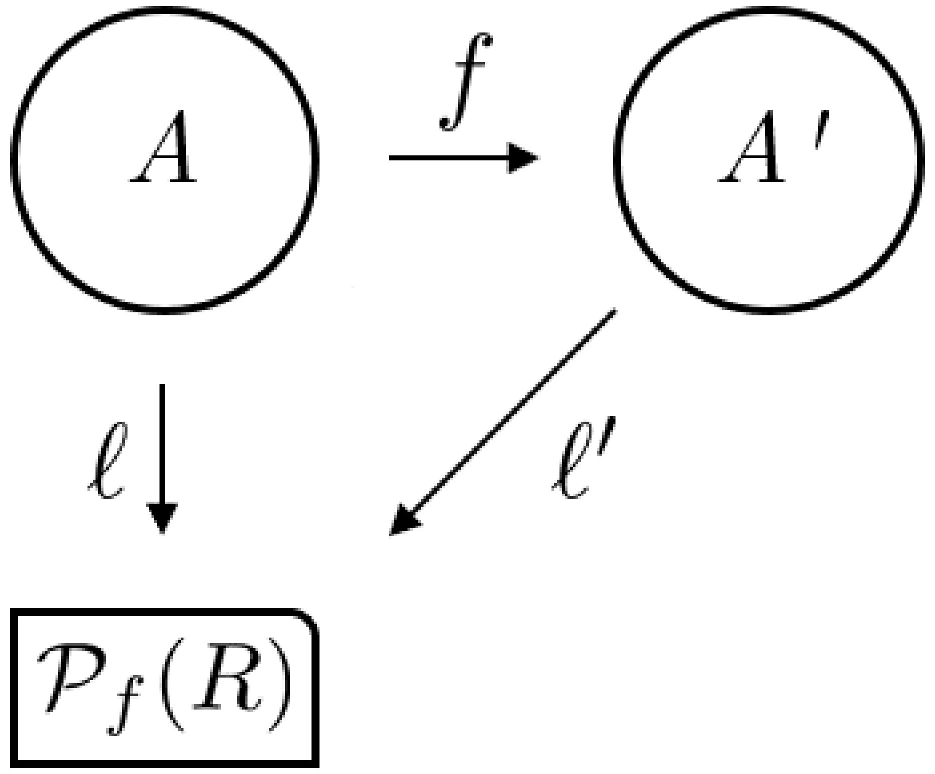

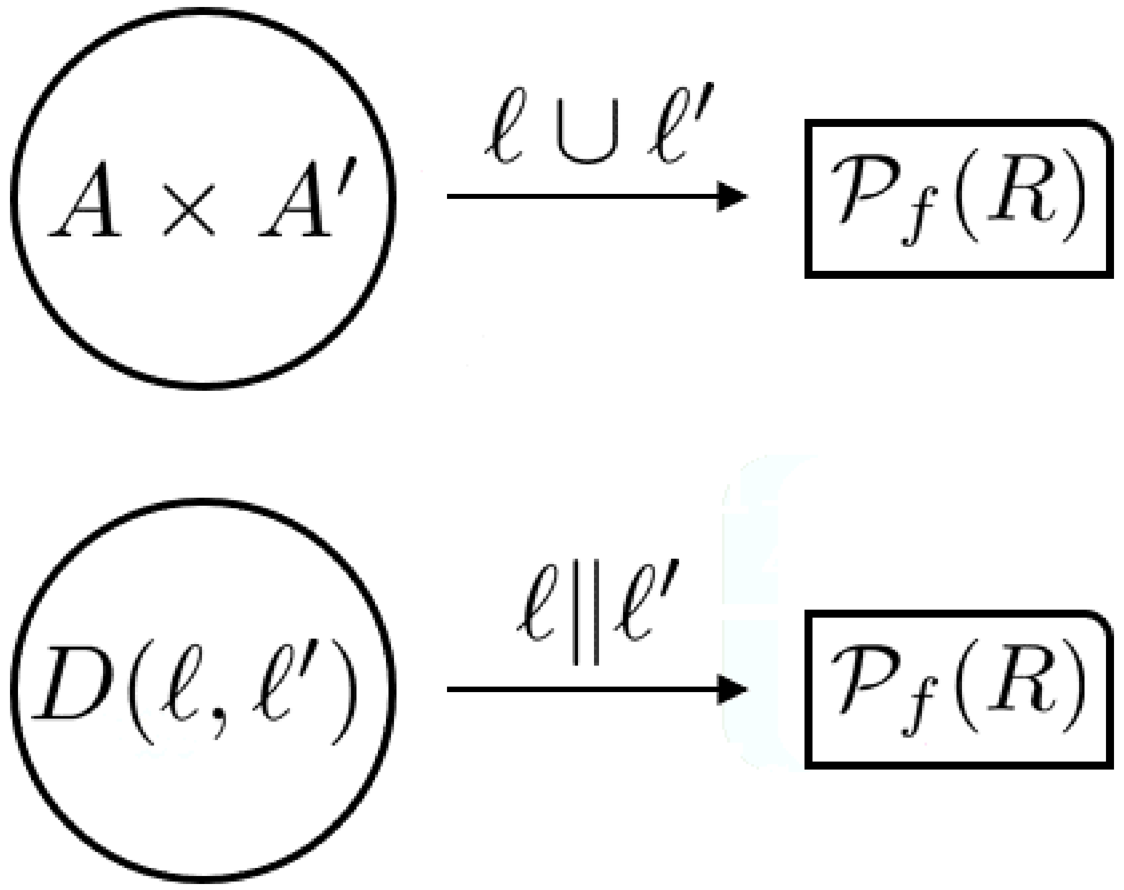

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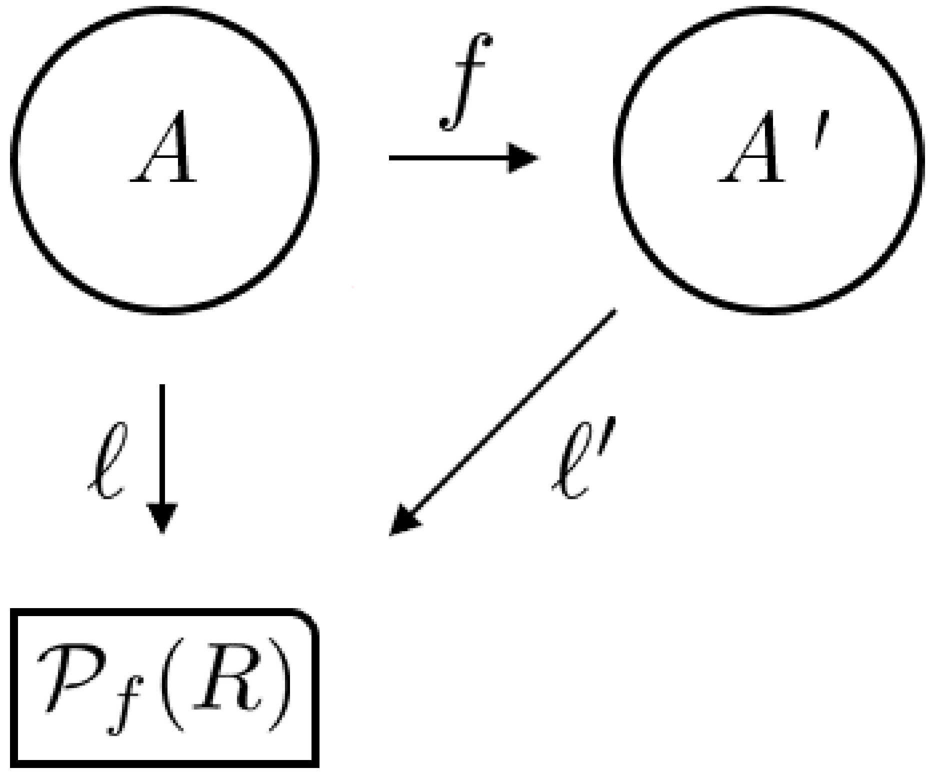

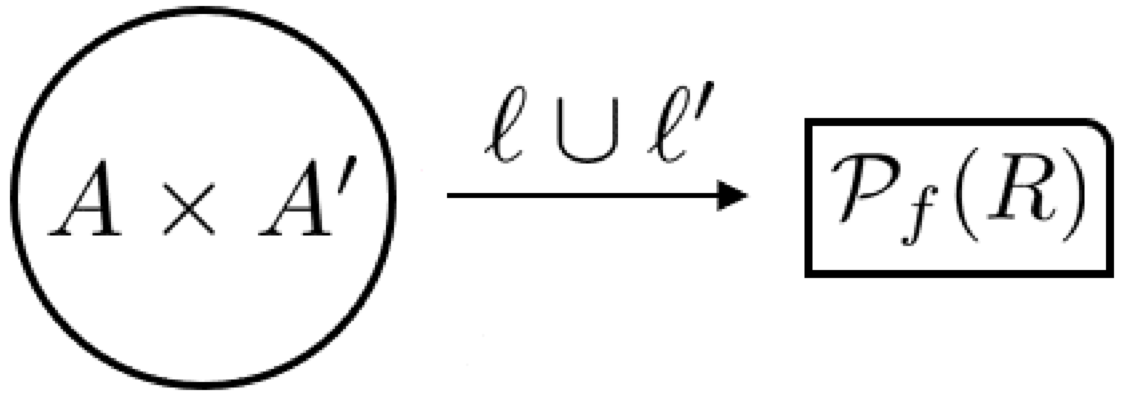

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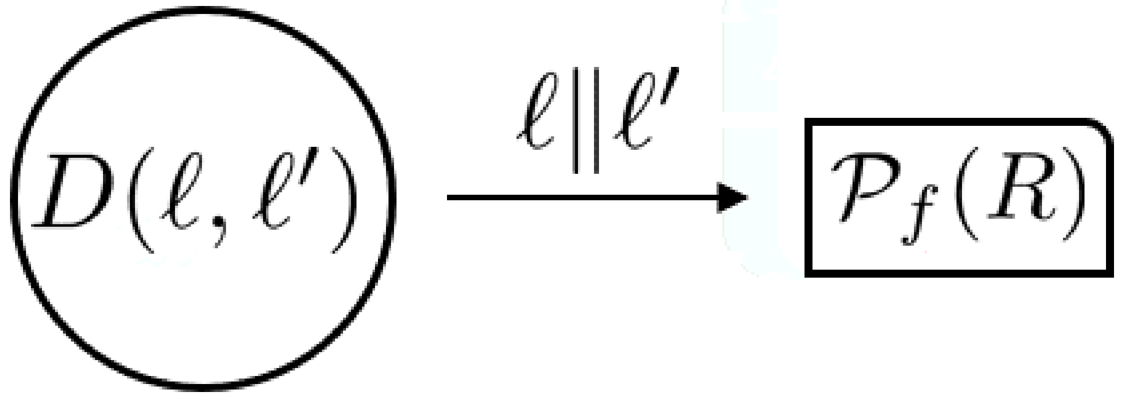

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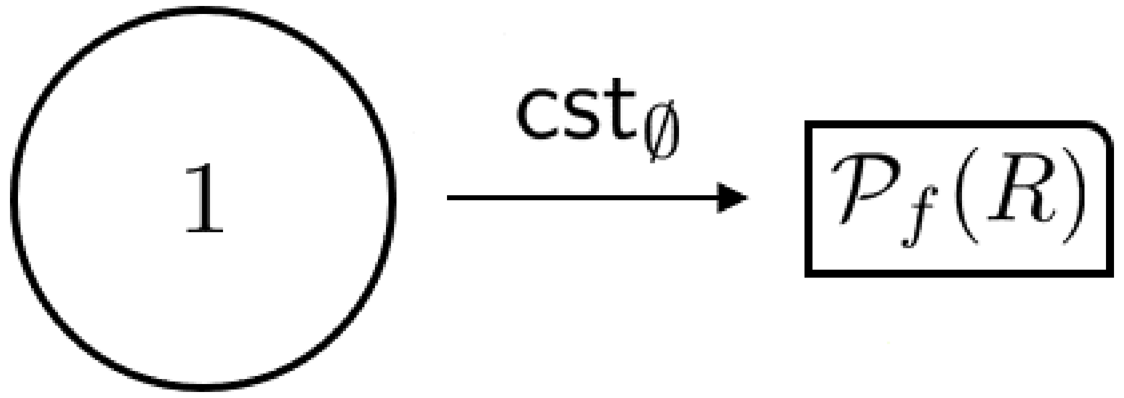

$\mathsf{cst}_{\emptyset}(\star) \coloneqq \emptyset$

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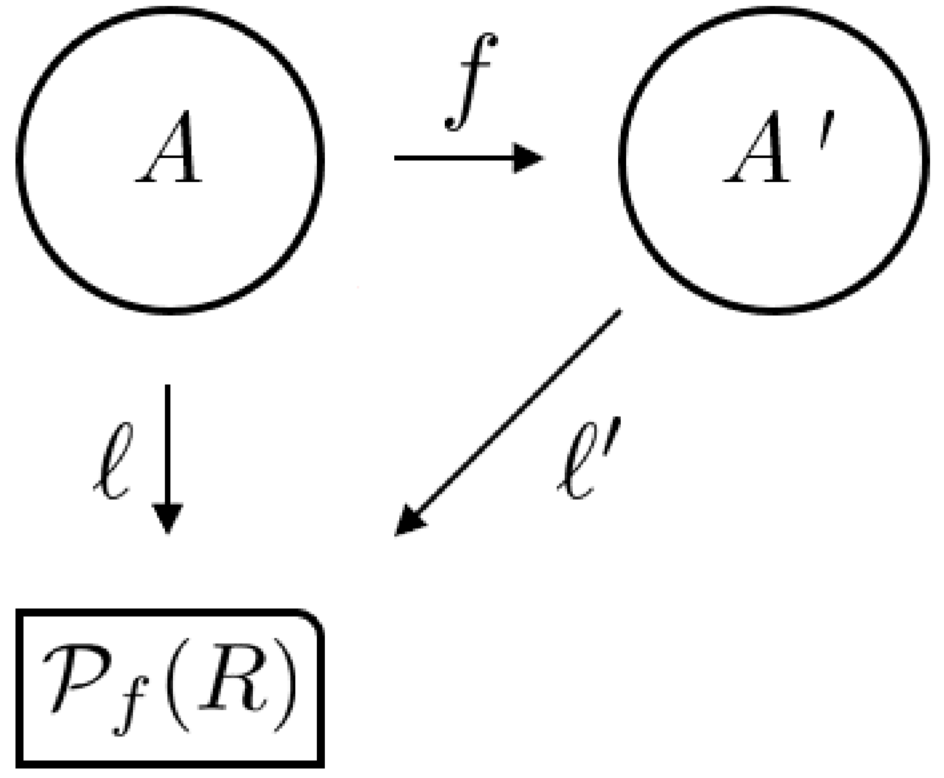

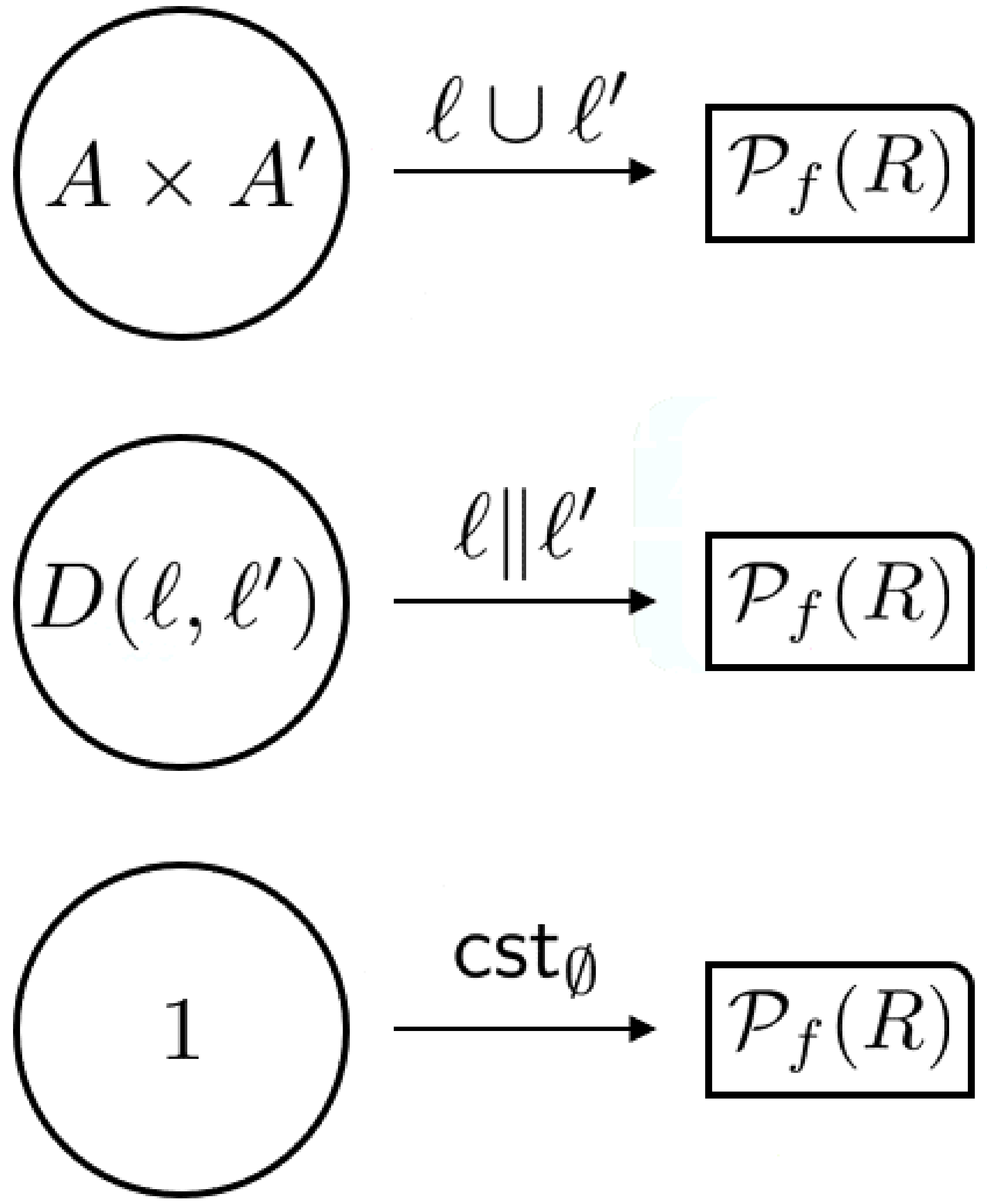

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$(\mathbf{Label}, \|, \mathsf{cst}_\emptyset, \cup, \mathsf{cst}_\emptyset)$ is duoidal

Category of Distinguished Subsets

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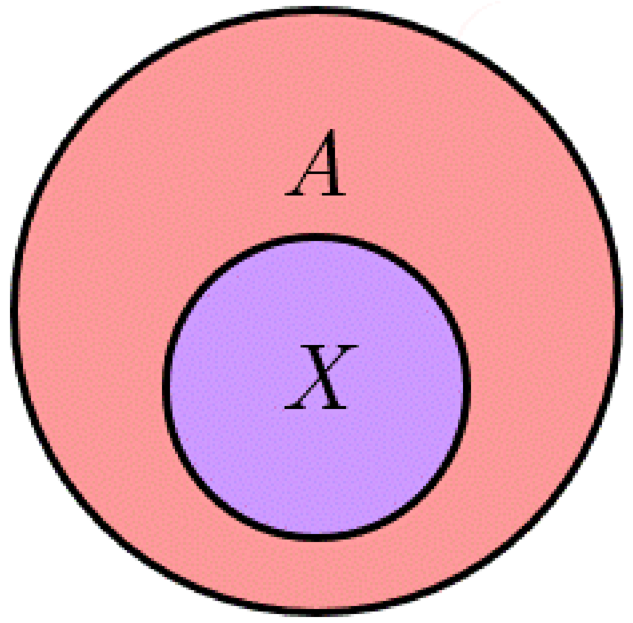

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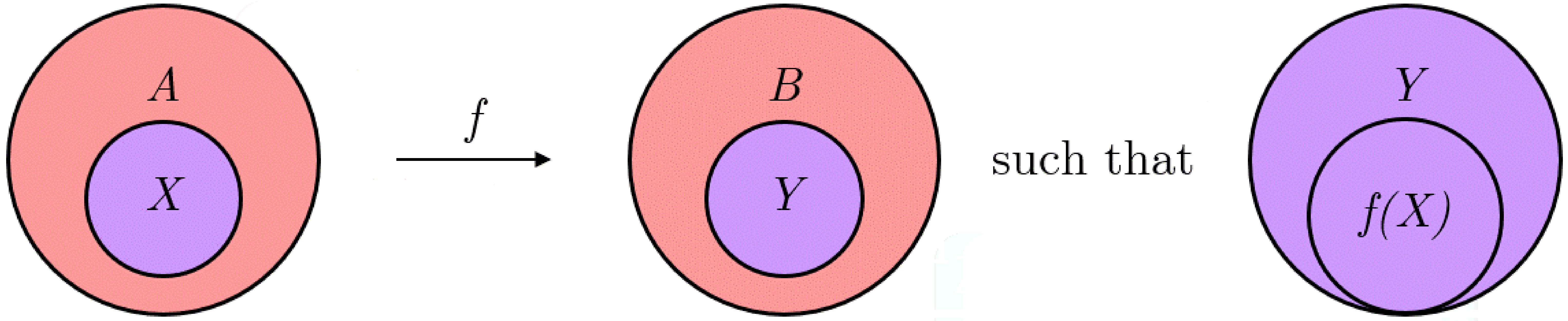

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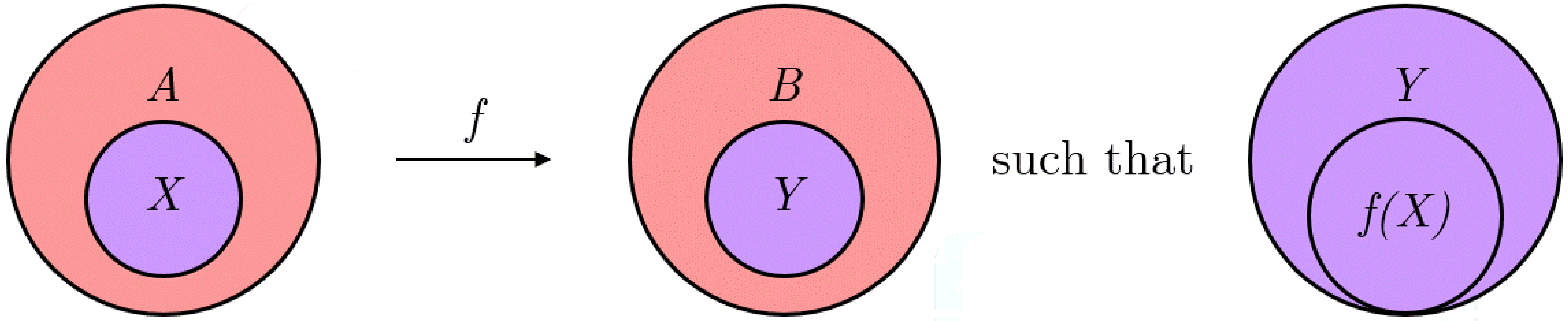

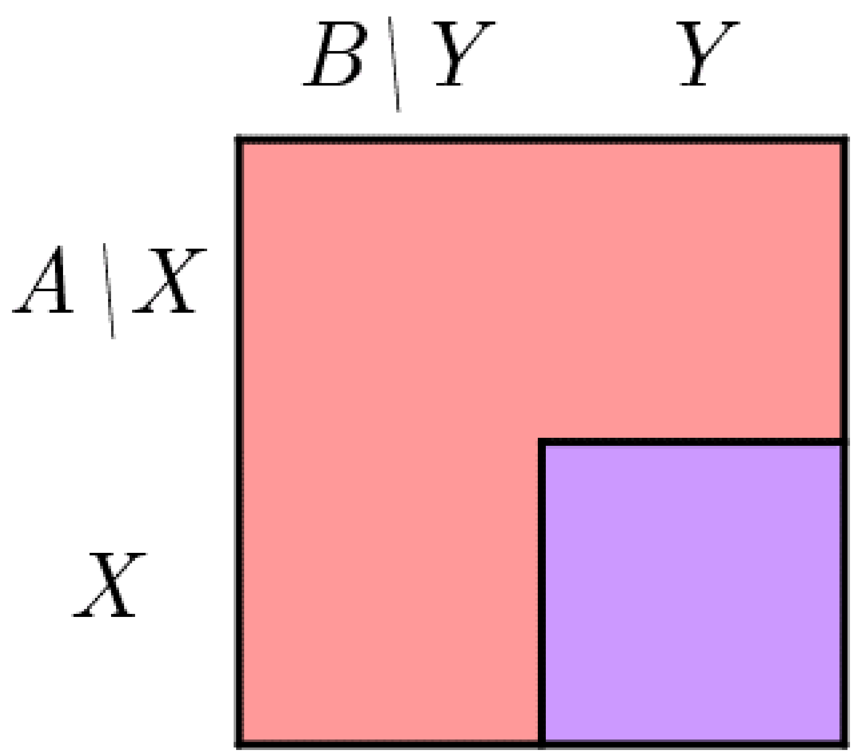

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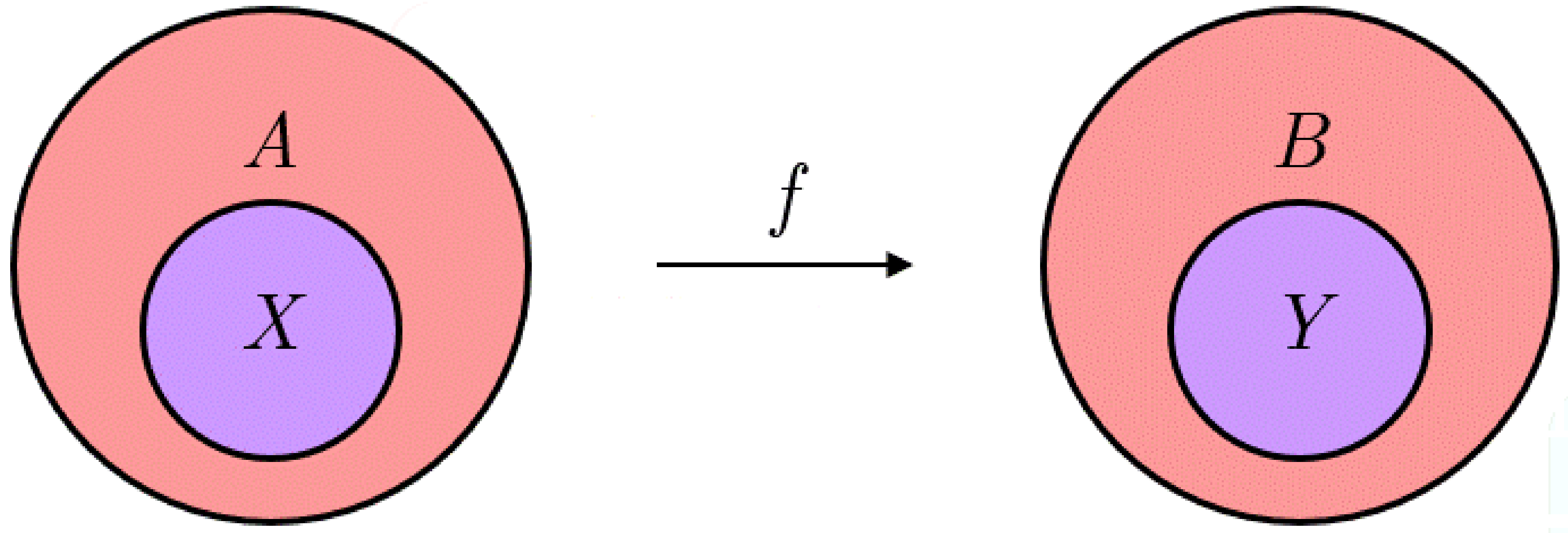
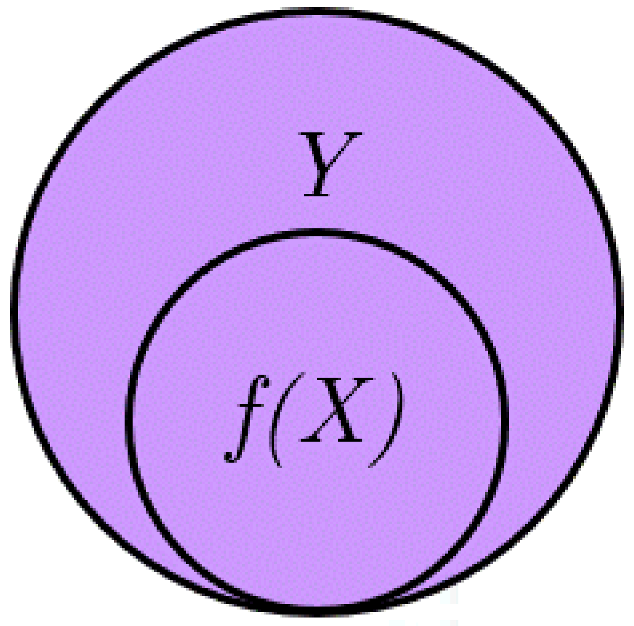
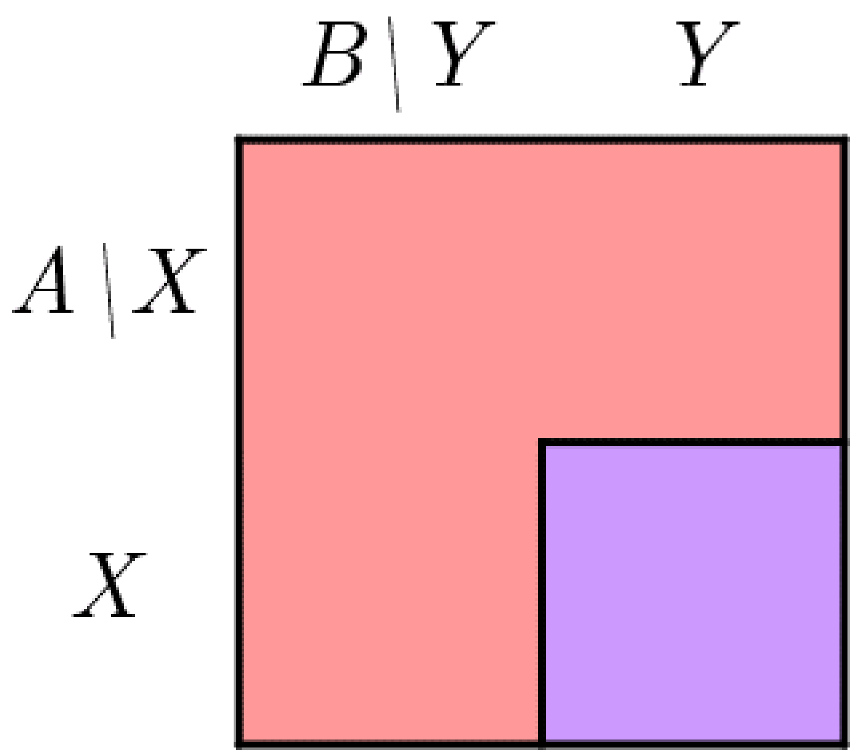
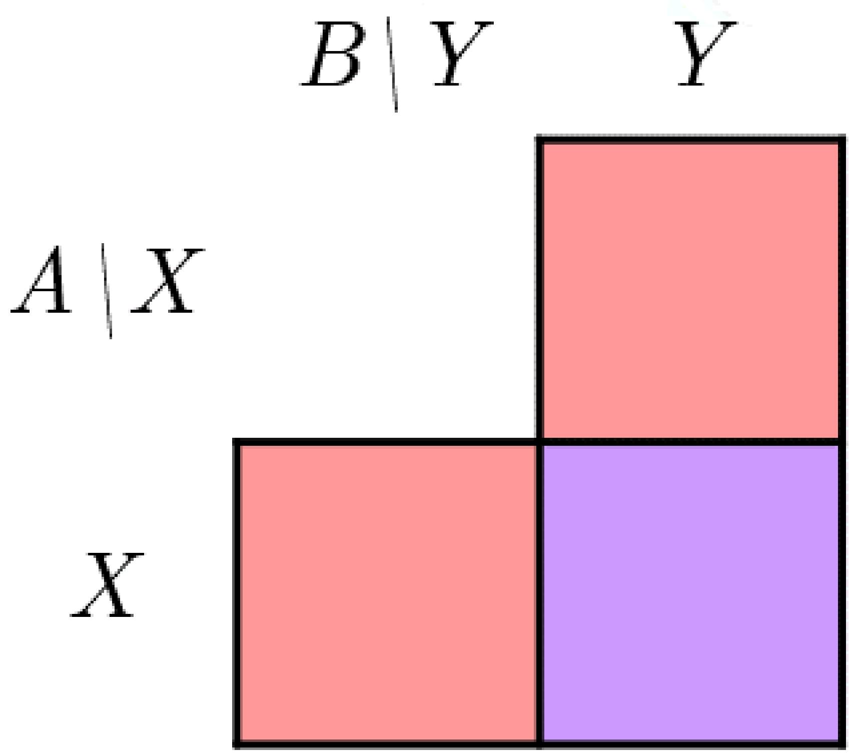


such that







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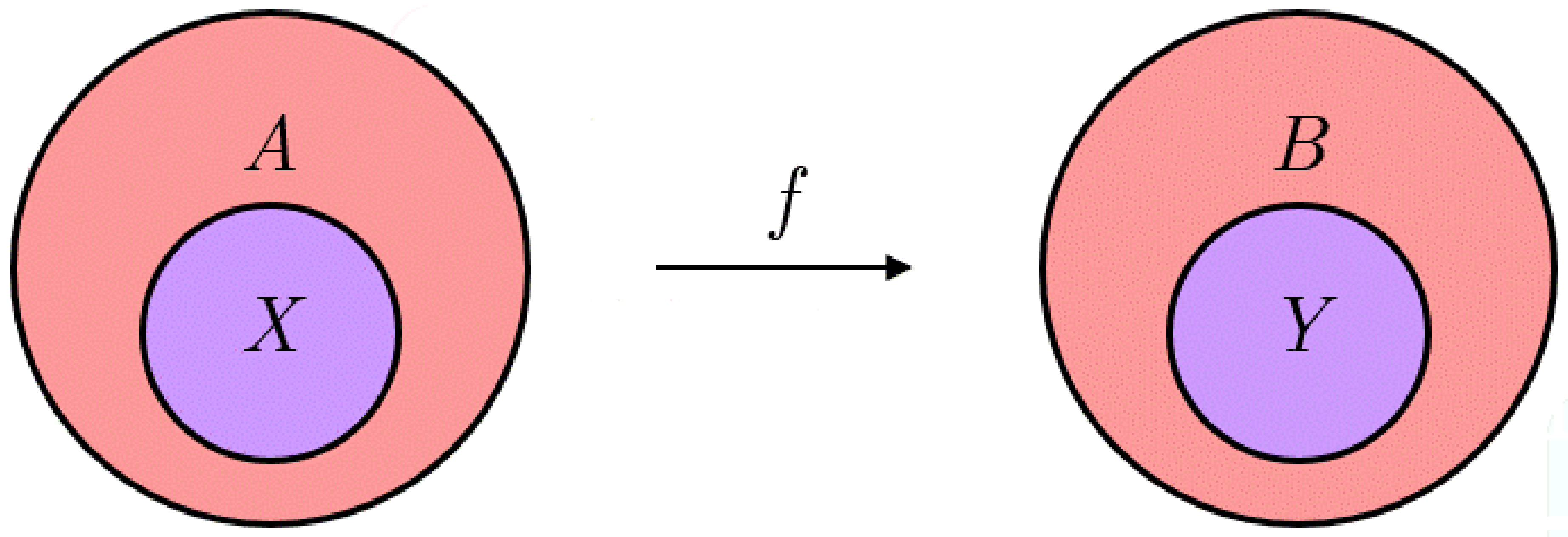

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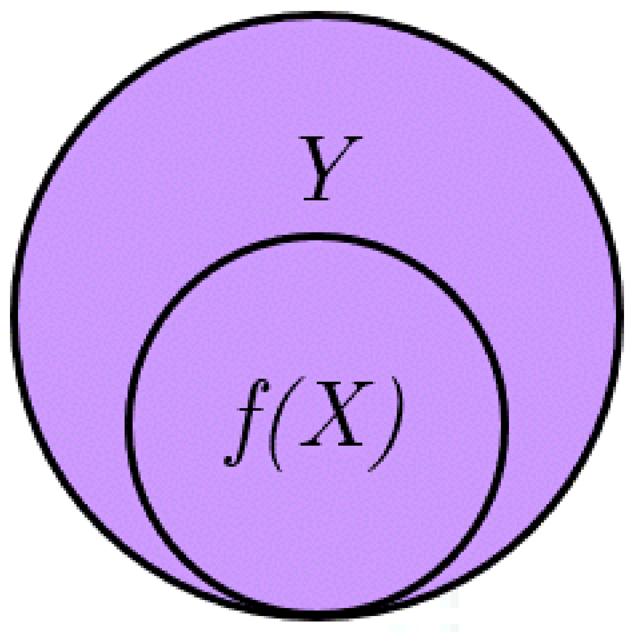

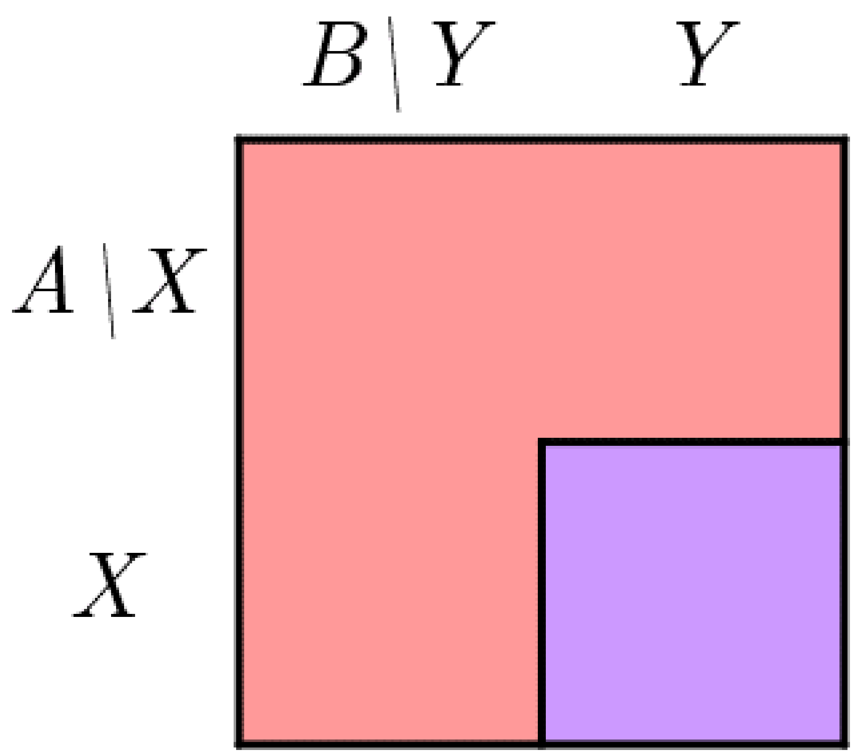

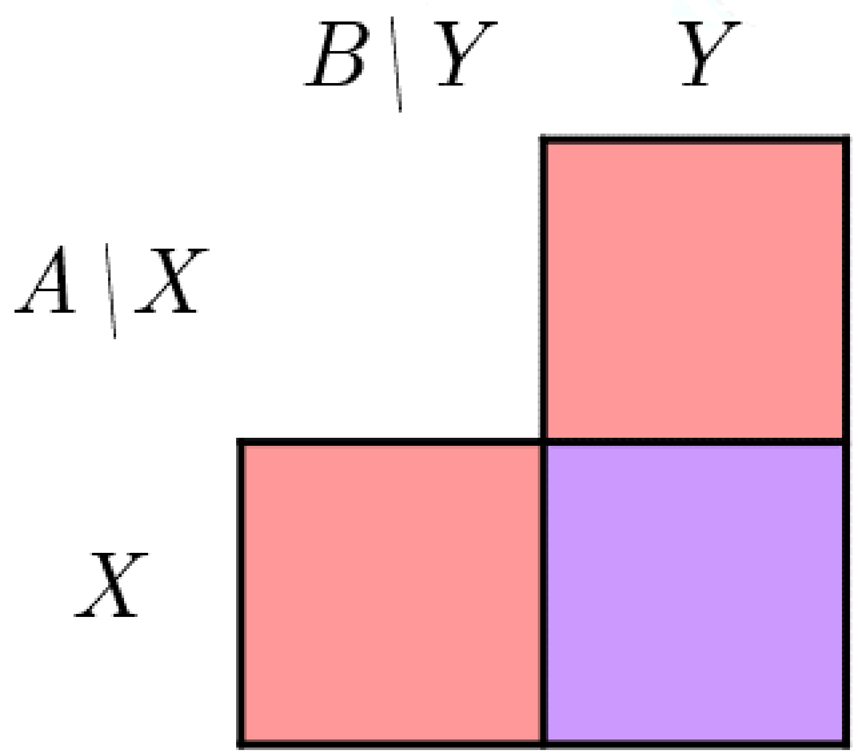

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$\big(\mathbf{Subset}, \otimes, (1,1), \times, (1,1)\big)$ is duoidal

Duoidally Enriched Freyd Categories

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Let $(\mathbf{V}, *, J, \circ, I)$ be a duoidal category and $(\mathbf{M}, \oplus, e)$ a monoidal category. A $\mathbf{V}$-*Freyd category over* $\mathbf{M}$ consists of

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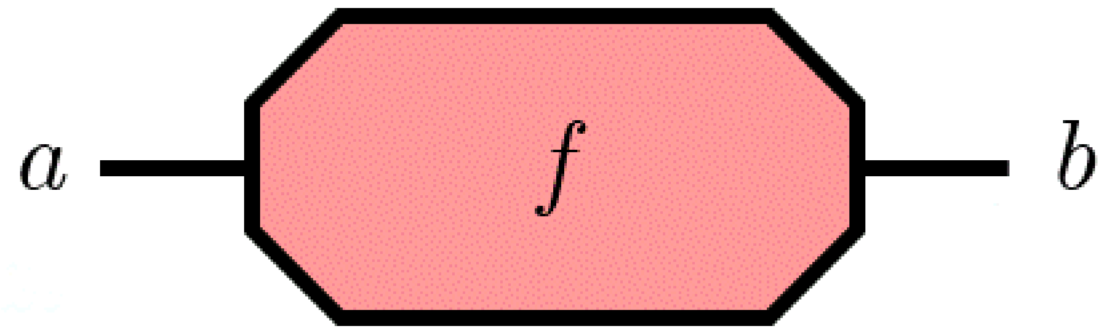
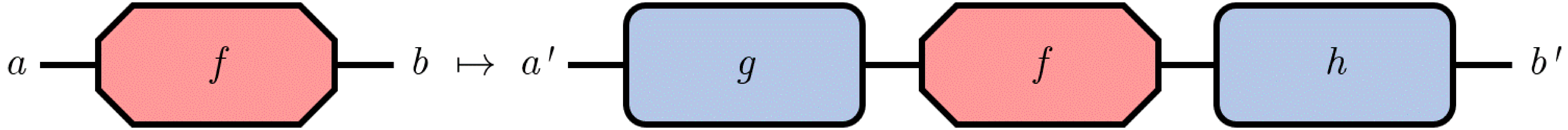
$\mathbf{C}\colon (a, b) \mapsto$



$\mathbf{C}(g, h)\colon$



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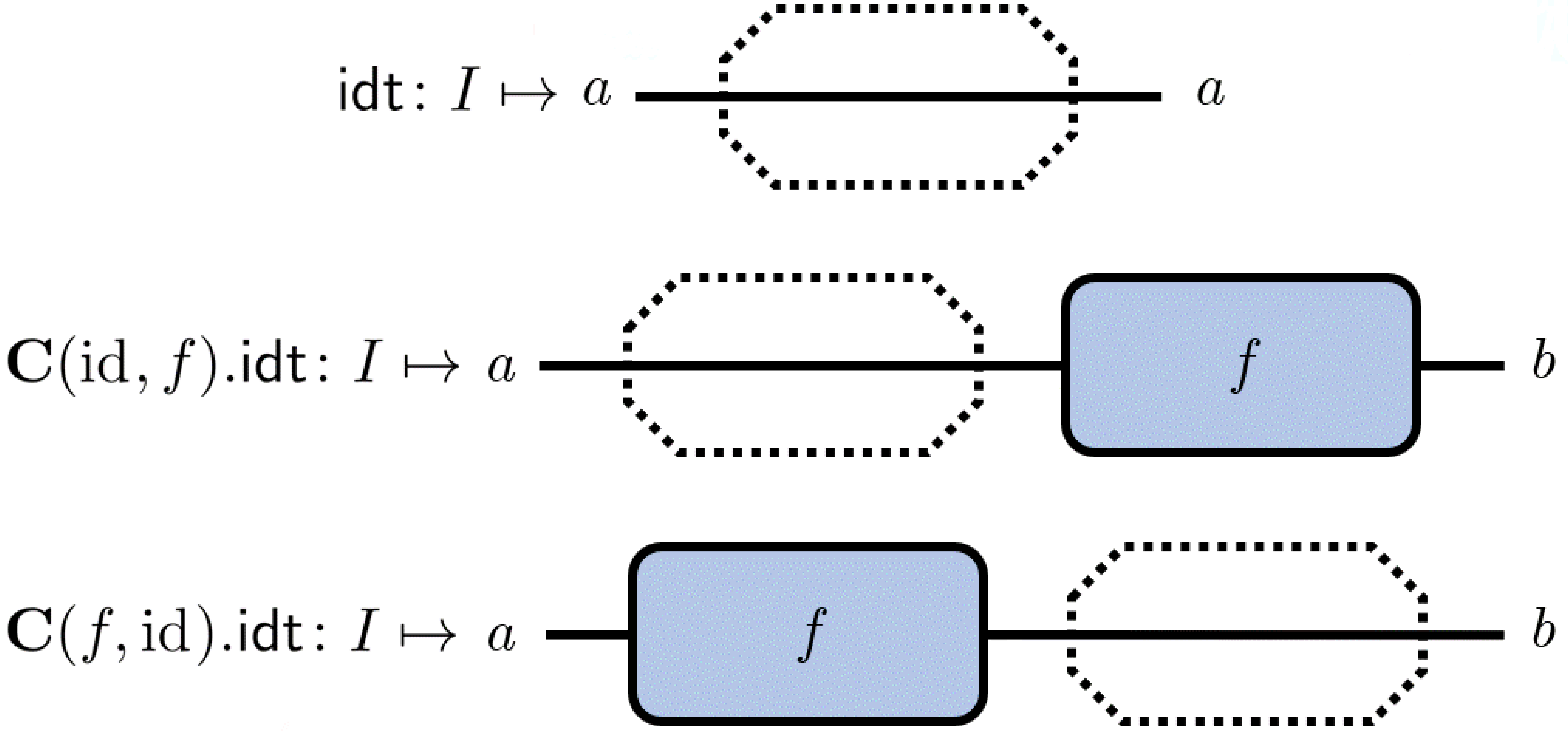

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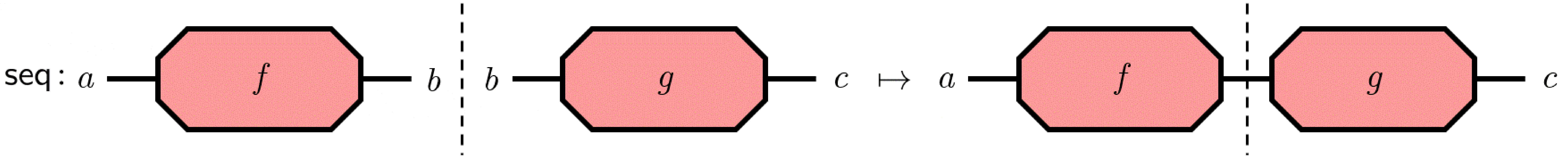
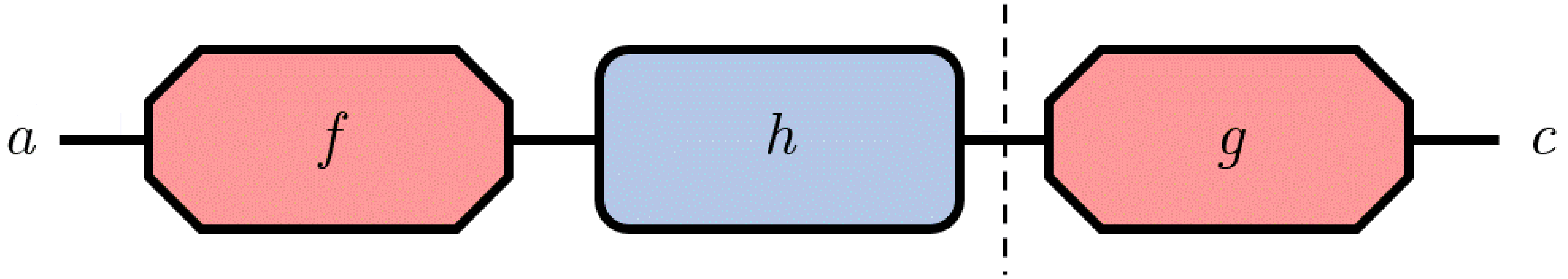
- an extranatural family $\mathsf{seq} \colon \mathbf{C}(a, b) \circ \mathbf{C}(b, c) \to \mathbf{C}(a, c)$, meaning seq is natural in a and c, and $\mathsf{seq}.(\mathrm{id} \circ \mathbf{C}(h, \mathrm{id})) = \mathsf{seq}.(\mathbf{C}(\mathrm{id}, h) \circ \mathrm{id})$





Duoidally Enriched Freyd Categories

Let $(\mathbf{V}, *, J, \circ, I)$ be a duoidal category and $(\mathbf{M}, \oplus, e)$ a monoidal category. A $\mathbf{V}$-*Freyd category over* $\mathbf{M}$ consists of

- a bifunctor $\mathbf{C}\colon \mathbf{M}^{\mathrm{op}} \times \mathbf{M} \to \mathbf{V}$
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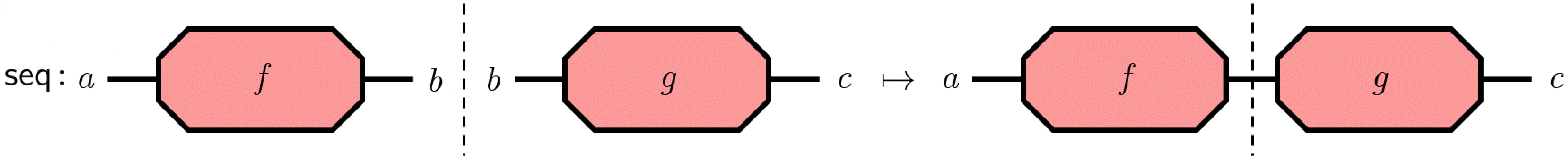

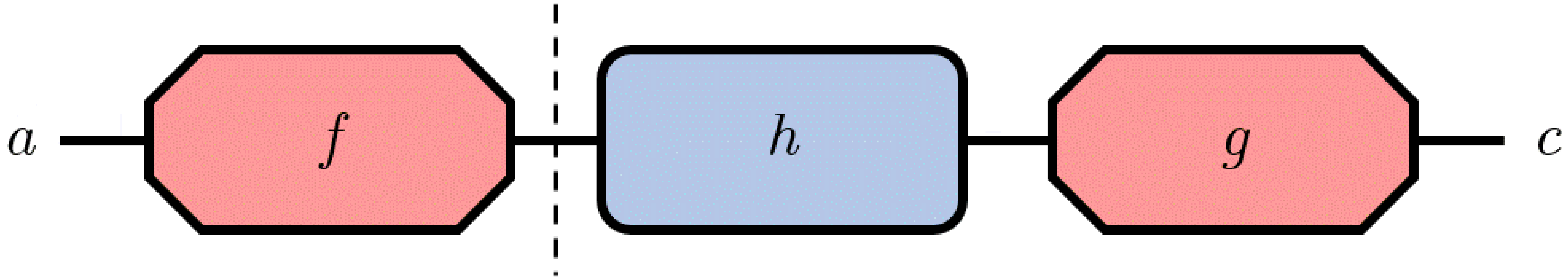

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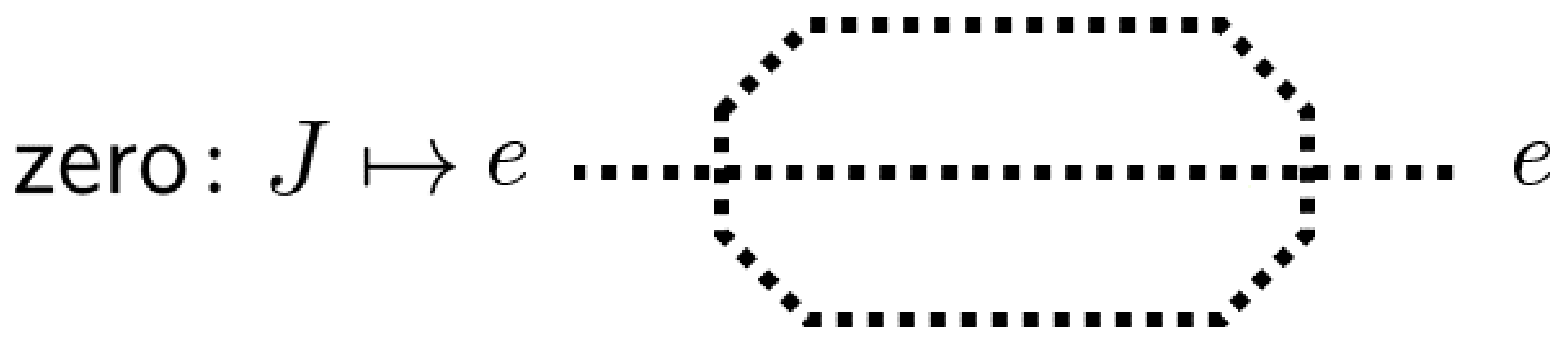
- an extranatural family $\mathsf{seq}\colon \mathbf{C}(a, b) \circ \mathbf{C}(b, c) \to \mathbf{C}(a, c)$, meaning seq is natural in a and c, and $\mathsf{seq}.(\mathrm{id} \circ \mathbf{C}(h, \mathrm{id})) = \mathsf{seq}.(\mathbf{C}(\mathrm{id}, h) \circ \mathrm{id})$
- a morphism $\mathsf{zero}\colon J \to \mathbf{C}(e, e)$



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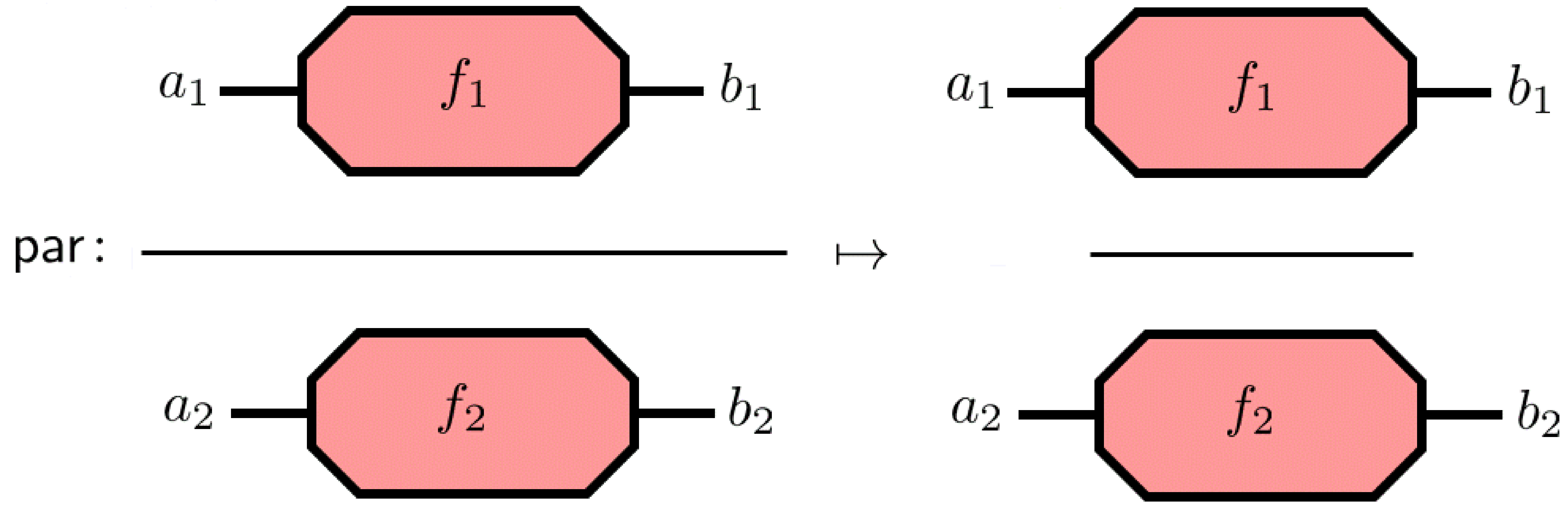
- a morphism $\mathsf{zero} \colon J \to \mathbf{C}(e, e)$
- a natural family $\mathsf{par} \colon \mathbf{C}(a_1, b_1) * \mathbf{C}(a_2, b_2) \to \mathbf{C}(a_1 \oplus a_2, b_1 \oplus b_2)$



Duoidally Enriched Freyd Categories

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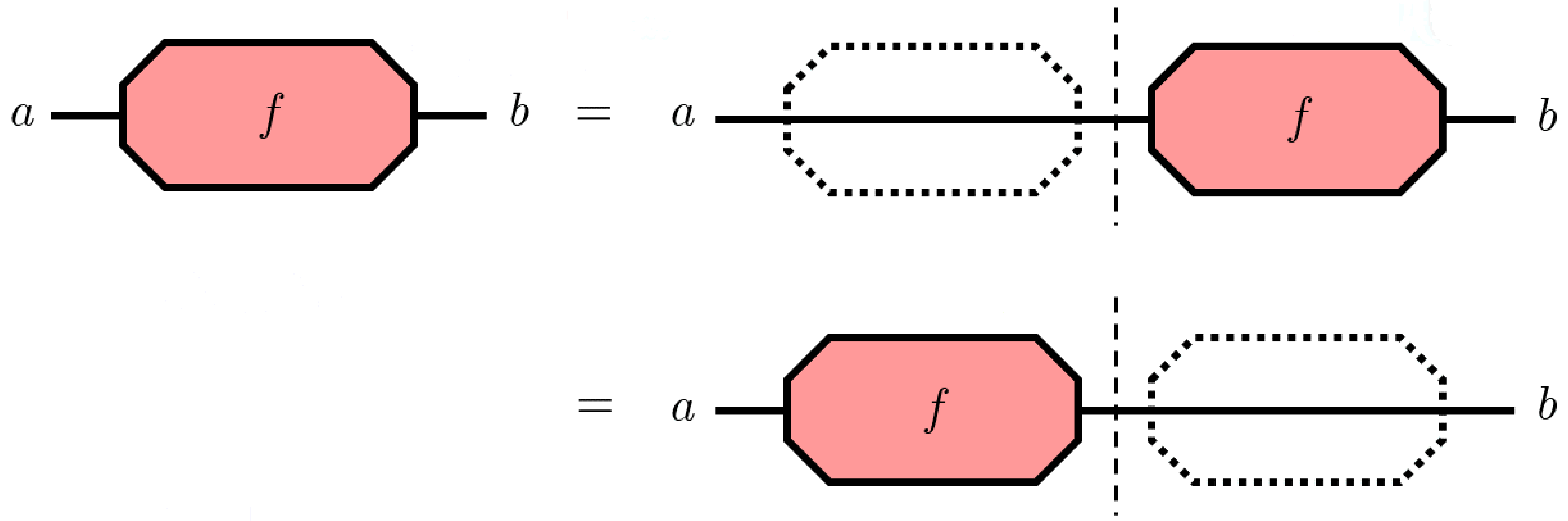
idt is the identity for seq, that is, $\mathsf{seq}.(\mathsf{idt} \circ \mathrm{id}) = \lambda$ and symmetrically



Duoidally Enriched Freyd Categories

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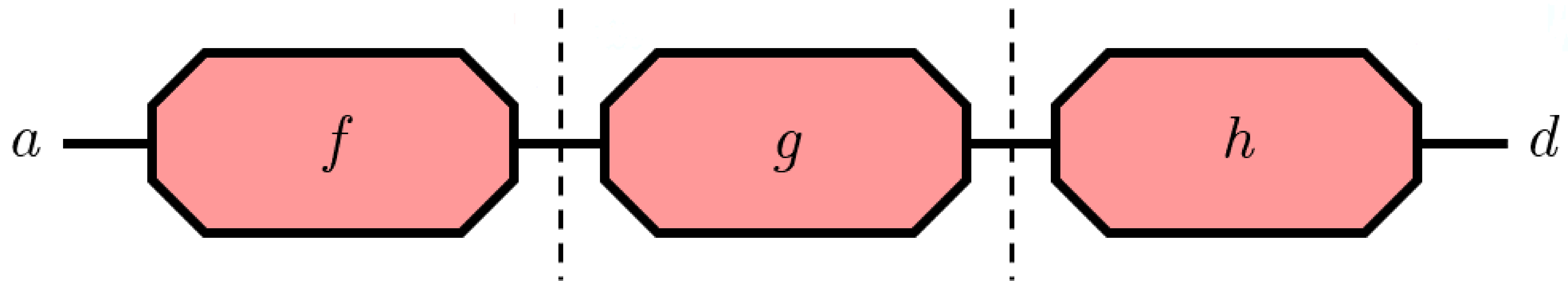
seq is associative, that is, $\mathsf{seq}.(\mathsf{seq} \circ \mathrm{id}) = \mathsf{seq}.(\mathrm{id} \circ \mathsf{seq}).\alpha$



Duoidally Enriched Freyd Categories

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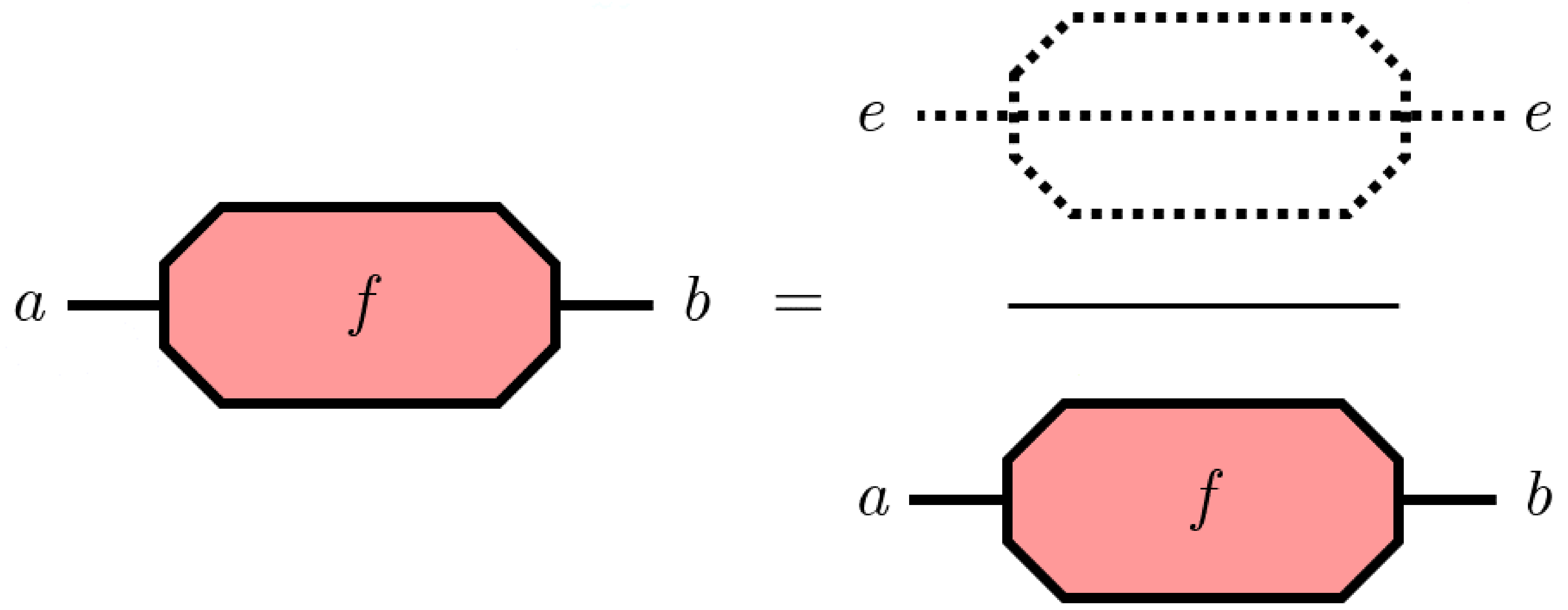
zero is the identity for par, that is, $\mathbf{C}(\lambda^{-1},\lambda).\mathsf{par}.(\mathsf{zero} * \mathrm{id}) = \lambda$ and sym.



Duoidally Enriched Freyd Categories

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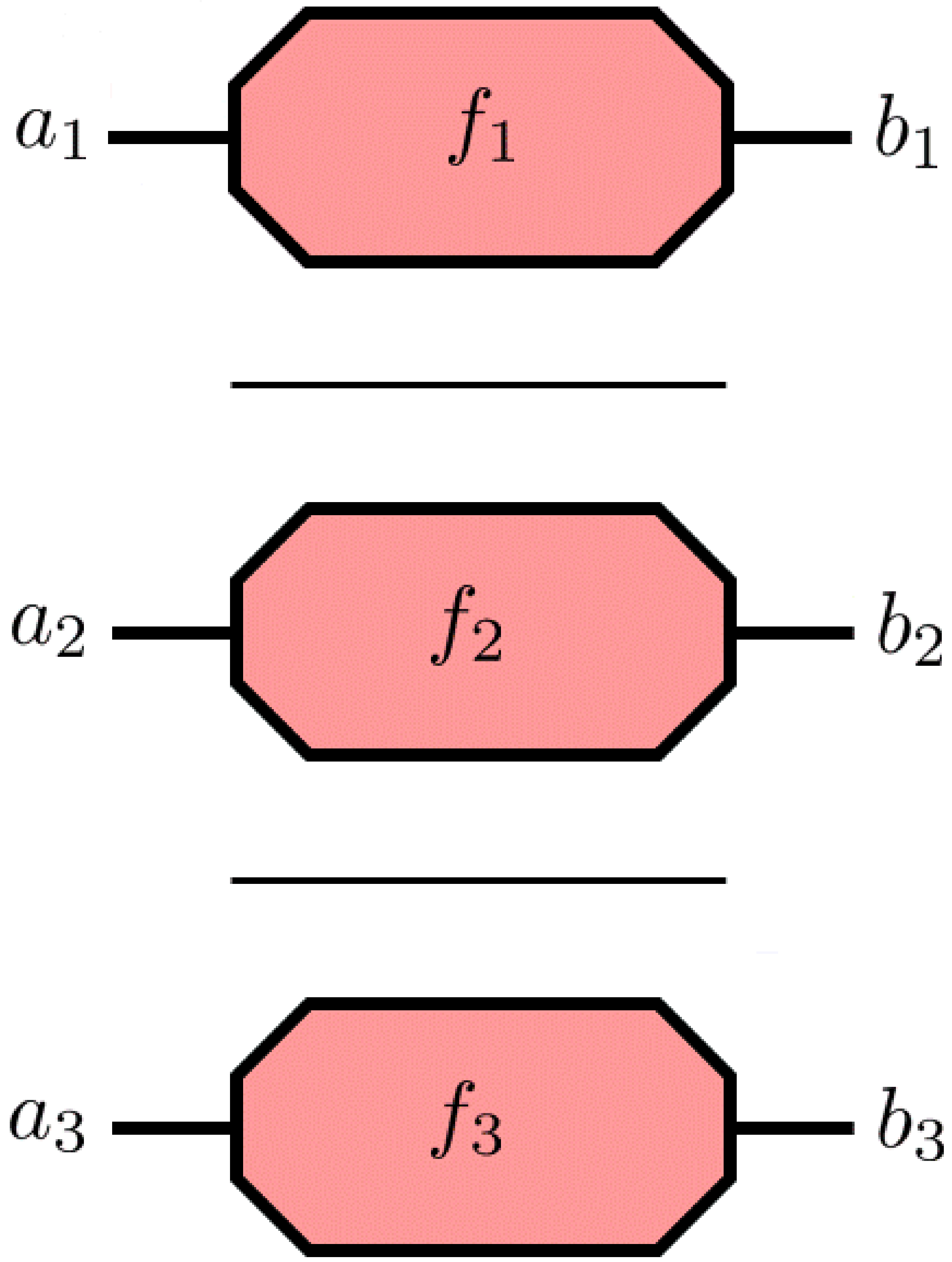

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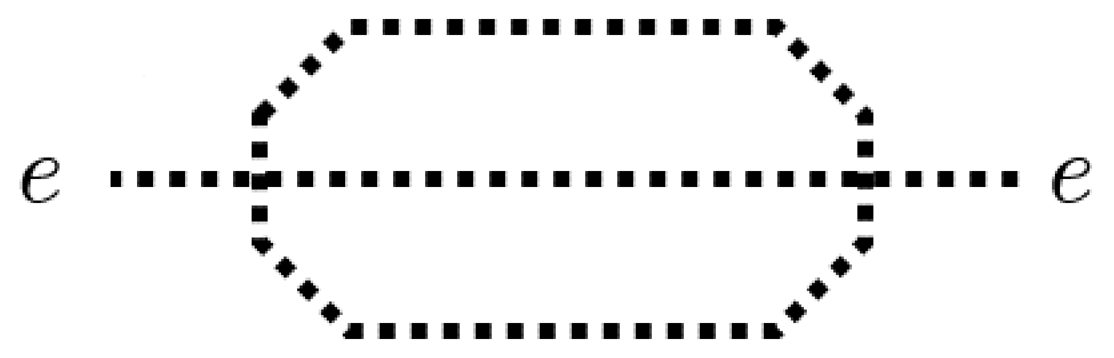
idt respects zero via $\mathsf{idt}.\epsilon = \mathsf{zero}$



$(A \circ B) * (C \circ D) \xrightarrow{\zeta} (A * C) \circ (B * D) \qquad J \xrightarrow{\Delta} J \circ J \qquad I * I \xrightarrow{\nabla} I \qquad J \xrightarrow{\epsilon} I$

Duoidally Enriched Freyd Categories

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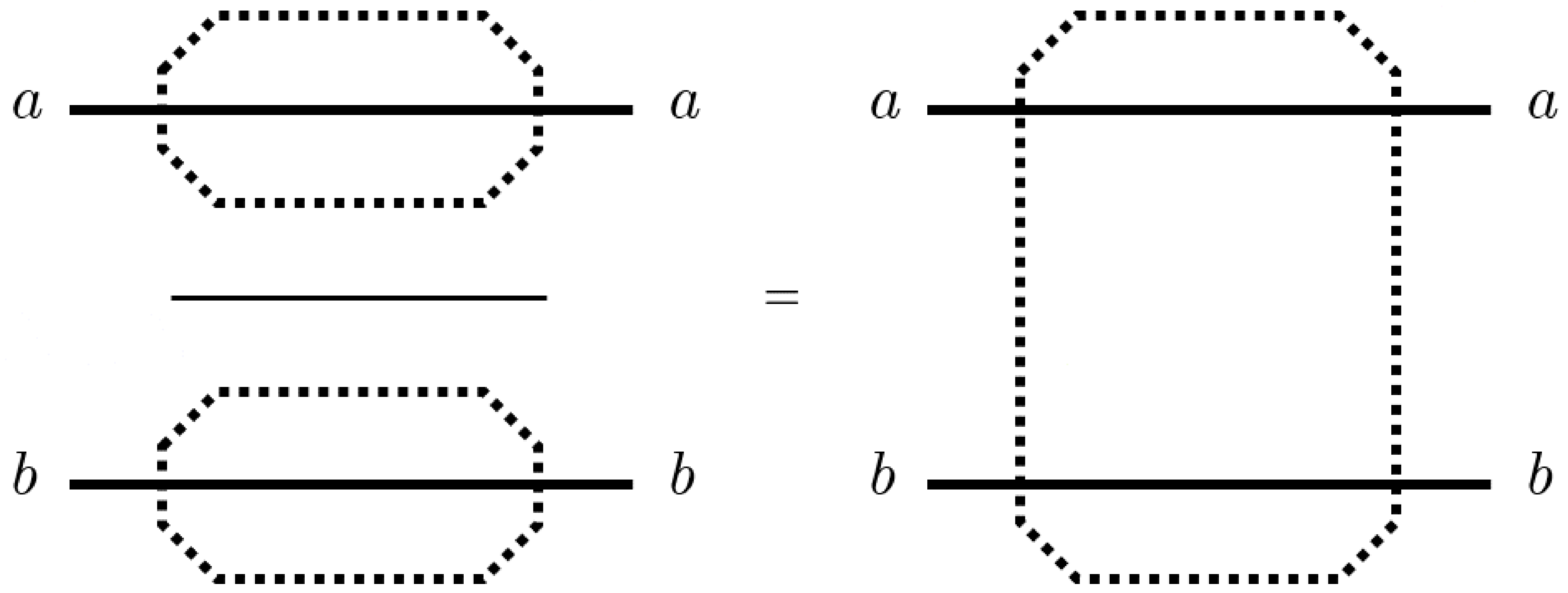
idt respects par via $\mathsf{idt}.\nabla = \mathsf{par}.(\mathsf{idt} * \mathsf{idt})$



$$* \frac{A \circ B}{C \circ D} \xrightarrow{\zeta} \overset{\circ}{\begin{array}{c:c} A & B \\ * & * \\ C & D \end{array}} \qquad J \xrightarrow{\Delta} \overset{\circ}{\begin{array}{c:c} J & J \end{array}} \qquad * \frac{I}{I} \xrightarrow{\nabla} I \qquad J \xrightarrow{\epsilon} I$$

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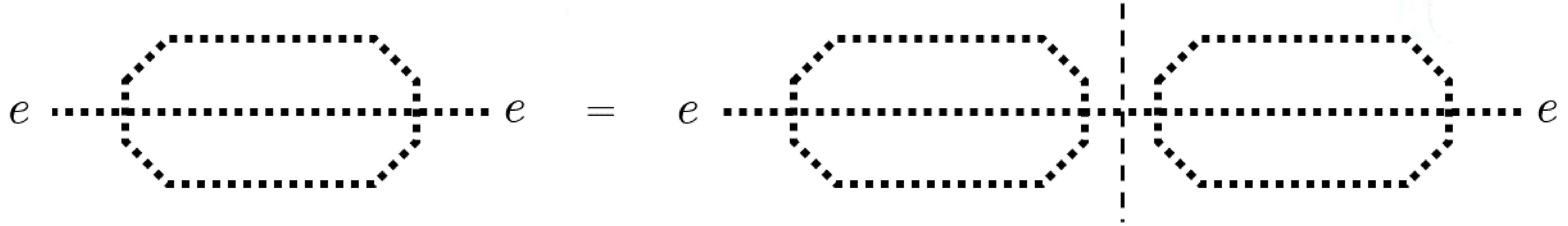

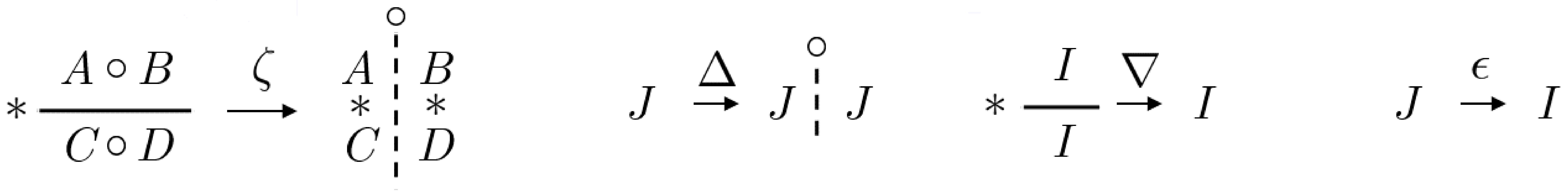

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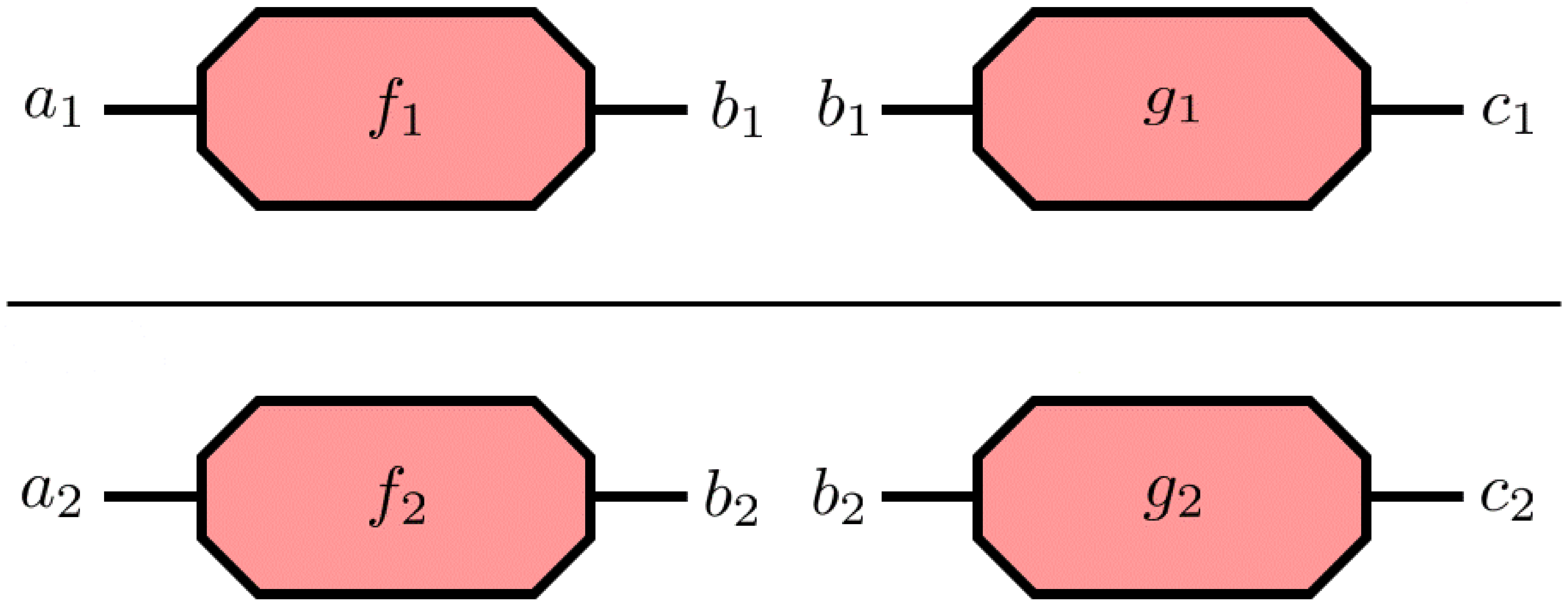

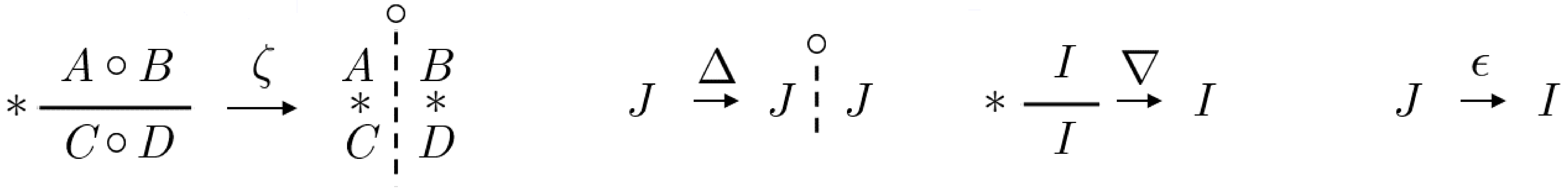

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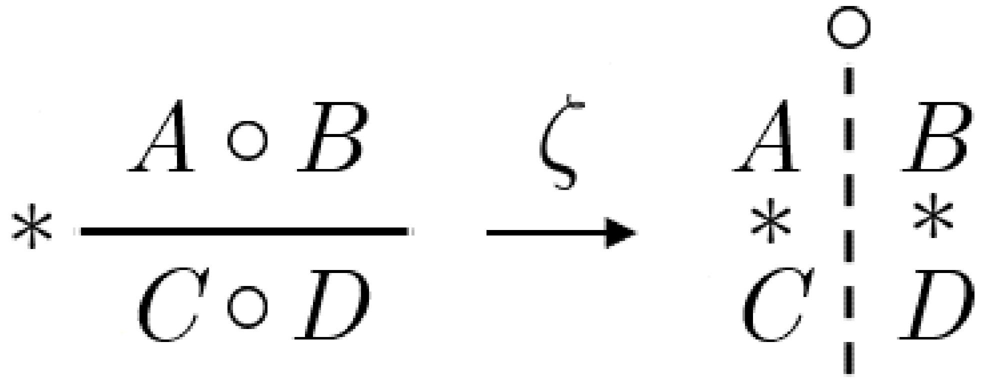

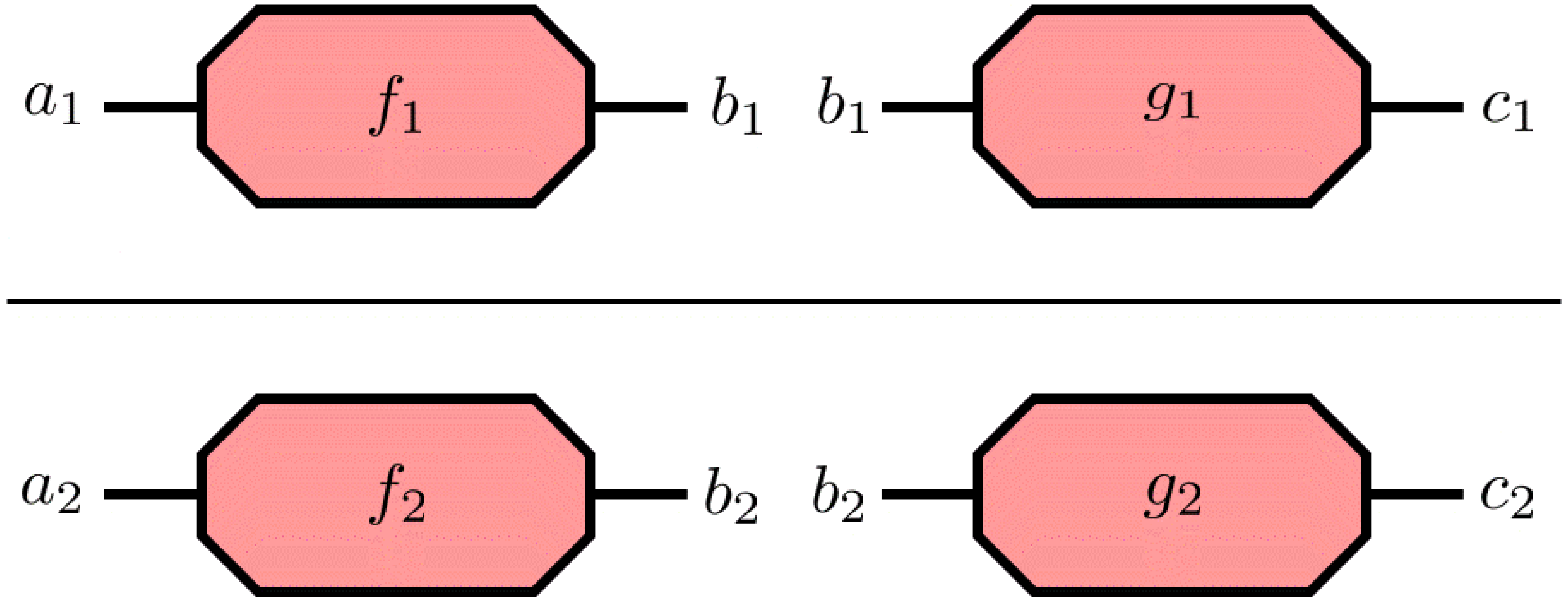

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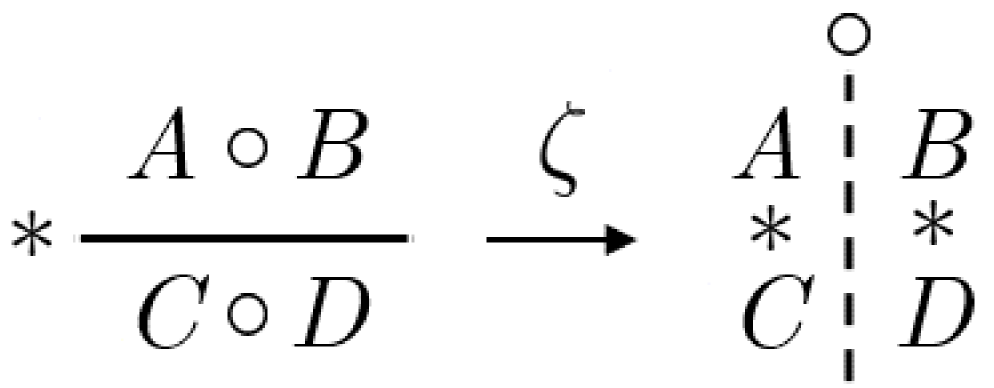

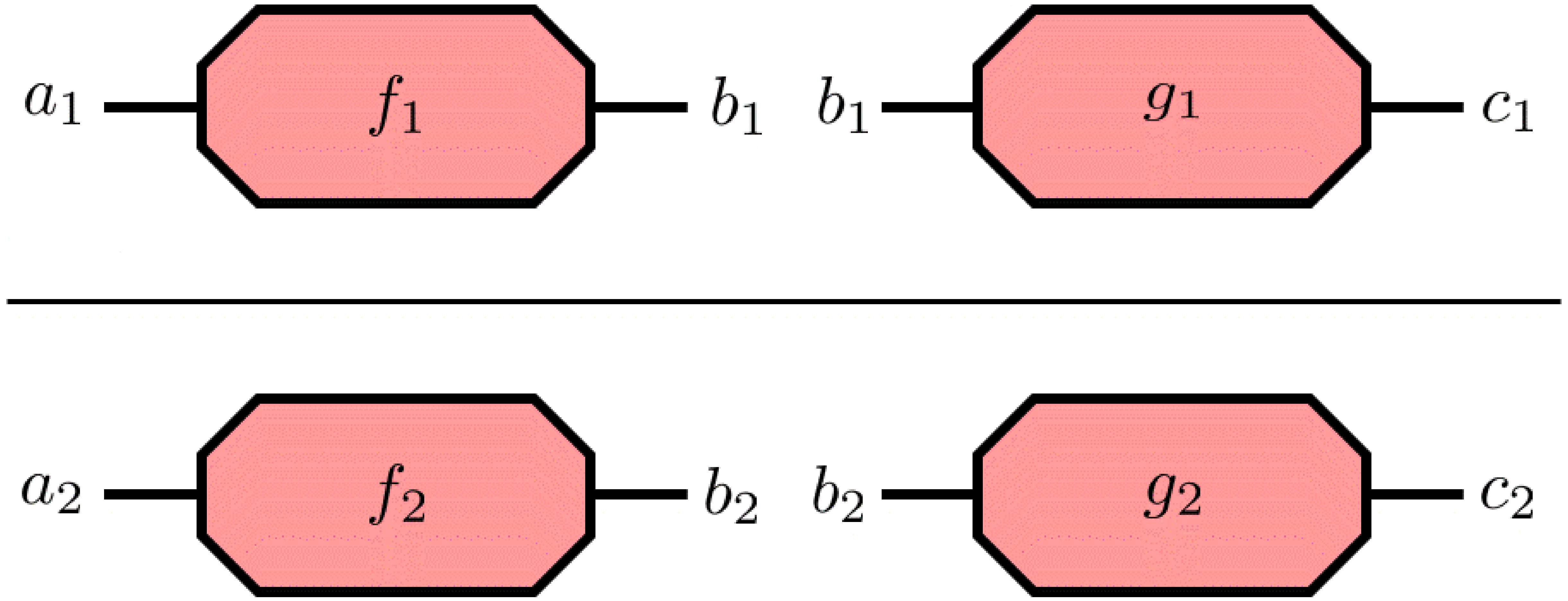

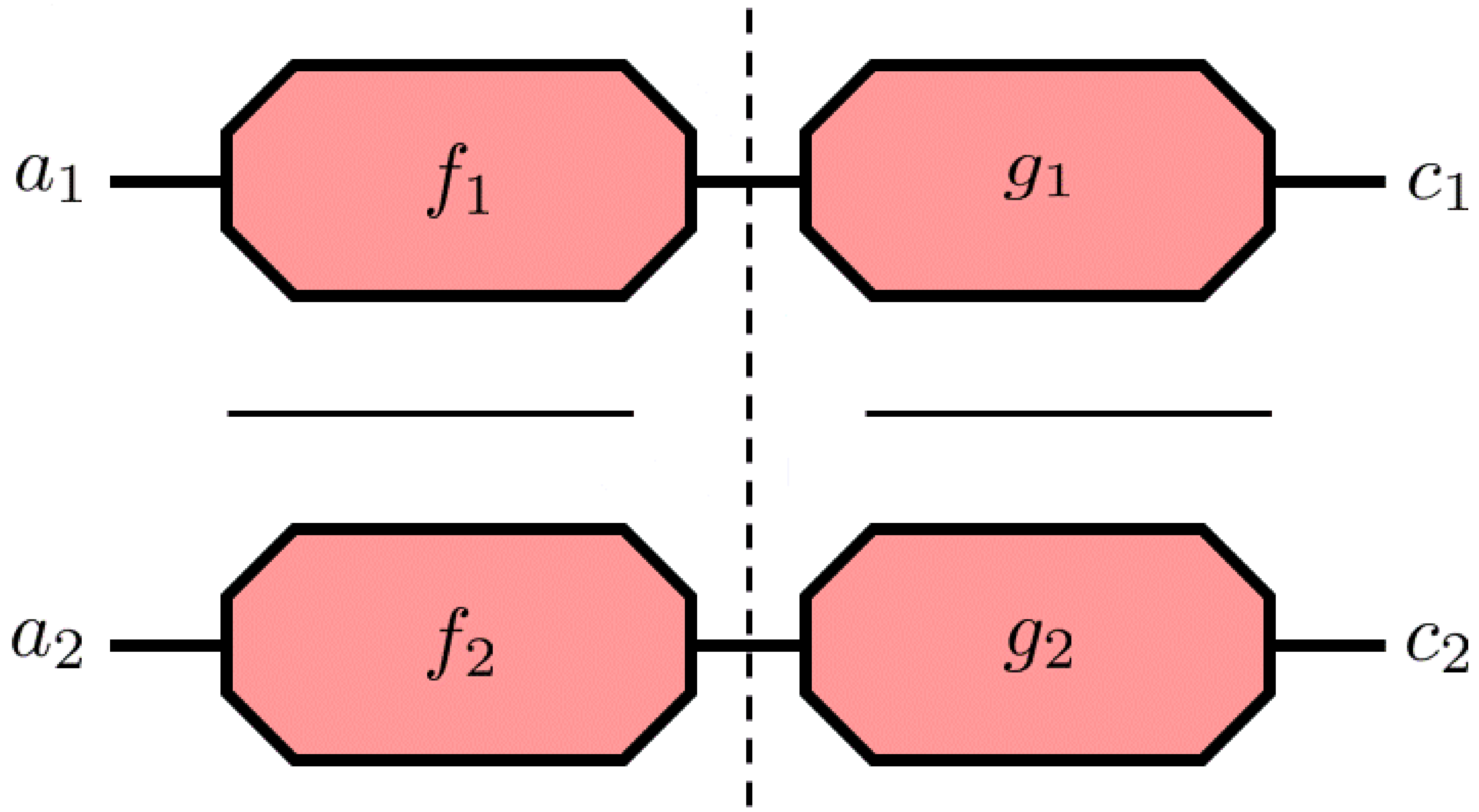

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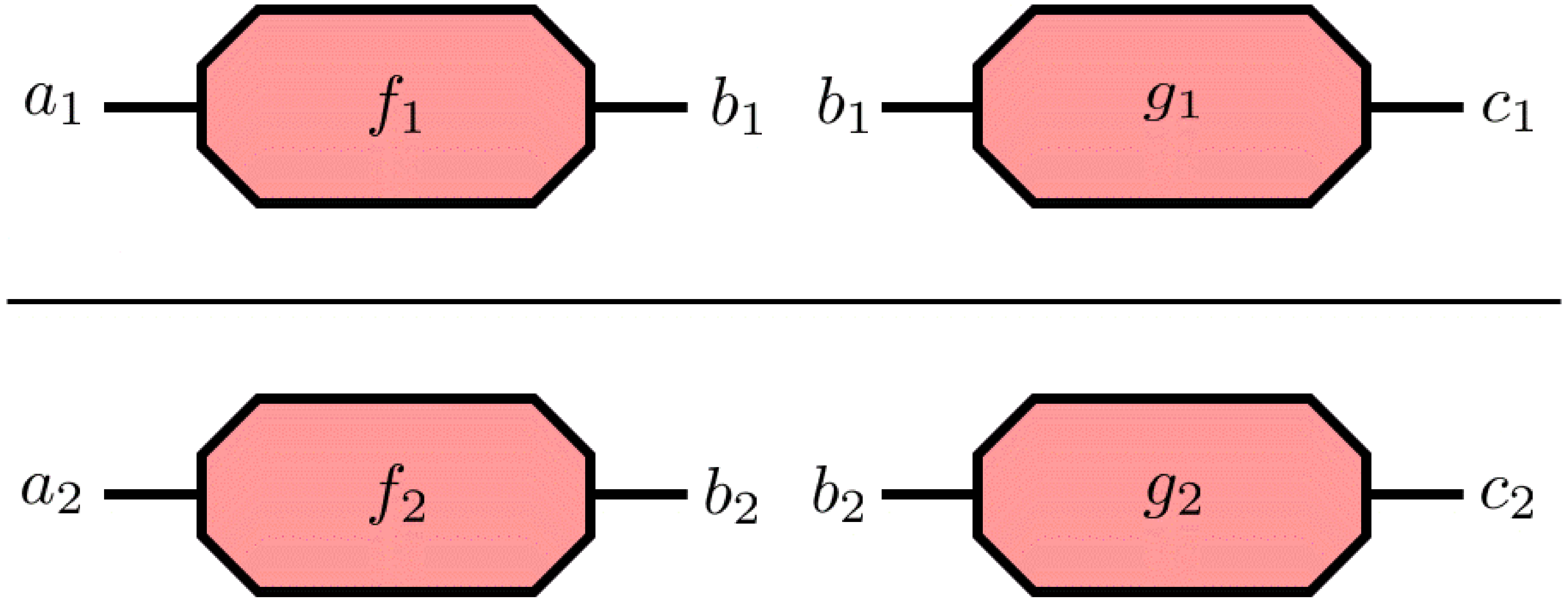
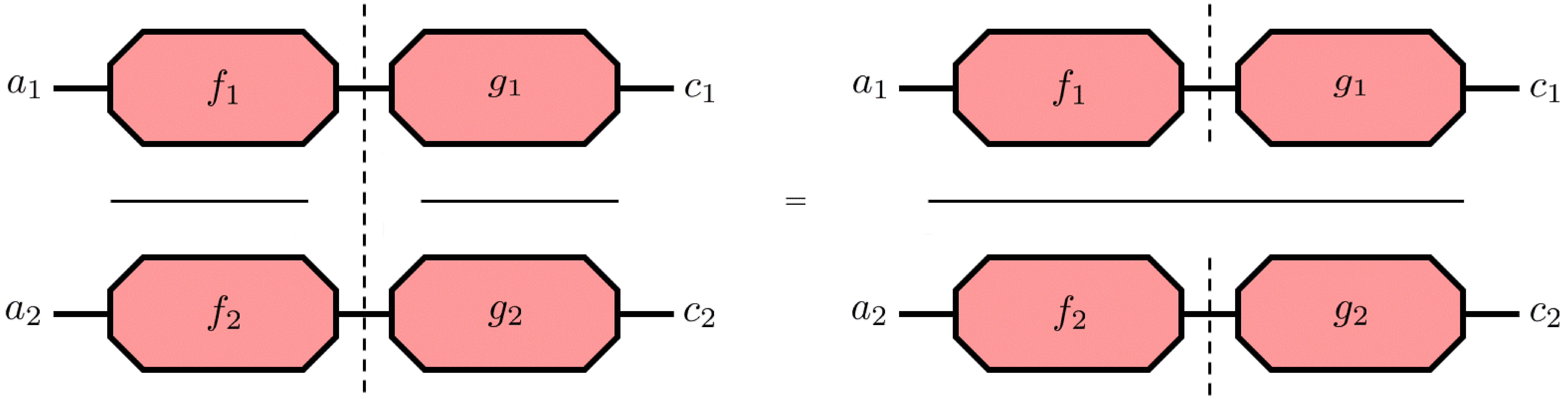
$$* \frac{A \circ B}{C \circ D} \xrightarrow{\zeta} \begin{matrix} A & \vdots & B \\ * & \vdots & * \\ C & \vdots & D \end{matrix}$$





Compound State

Compound State

$R \coloneqq \{\mathbb{B}, \mathbb{Z}, \ldots\}$ $\mathcal{P}_f(R) = \{\emptyset, \{\mathbb{B}\}, \{\mathbb{Z}\}, \{\mathbb{B}, \mathbb{Z}\}, \ldots\}$

$\mathbf{V} = (\mathbf{Label}, \|, \mathsf{cst}_\emptyset, \cup, \mathsf{cst}_\emptyset)$ $\mathbf{M} = (\mathbf{Set}, \times, 1)$

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$$\mathbf{C}(a,b) \coloneqq \left(\sum_{Q \in \mathcal{P}_f(R)} \mathbf{Set}(\Pi_Q \times a, \Pi_Q \times b) \right) \xrightarrow{\ell} \mathcal{P}_f(R)$$

$$(Q, f) \mapsto Q$$

$\Pi_Q \coloneqq \Pi_{x \in Q} x$ for $Q \in \mathcal{P}_f(R)$, e.g. $\Pi_{\{\mathbb{B},\mathbb{Z}\}} = \mathbb{B} \times \mathbb{Z}$

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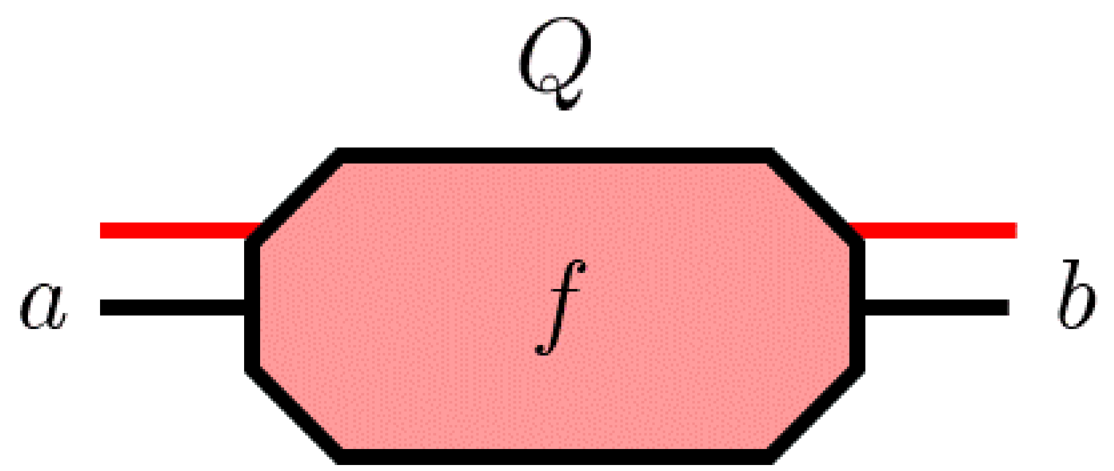
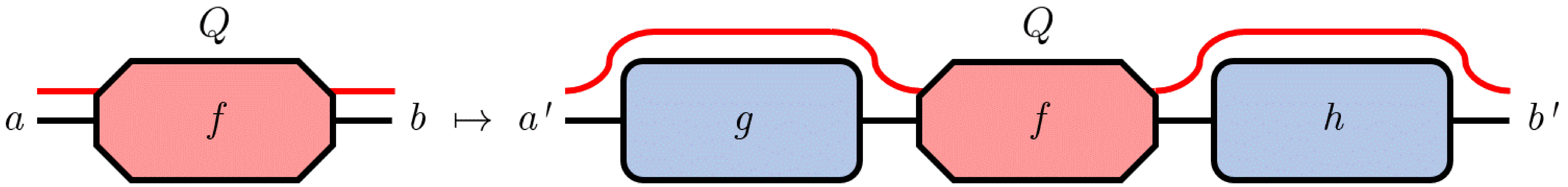
$\mathbf{C}\colon (a, b) \mapsto$



$\mathbf{C}(g, h)\colon$



Compound State

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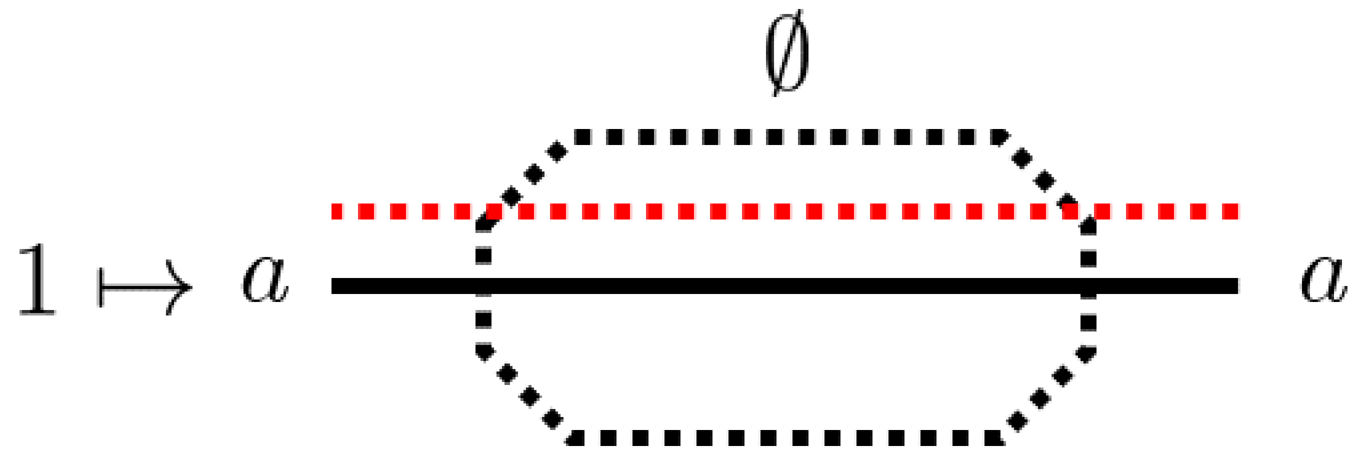
$$\mathbf{C}(a,b) \coloneqq \left(\sum_{Q \in \mathcal{P}_f(R)} \mathbf{Set}(\Pi_Q \times a, \Pi_Q \times b) \right) \xrightarrow{\ell} \mathcal{P}_f(R)$$

$$\mathsf{idt} \colon \mathsf{cst}_\emptyset \to \mathbf{C}(a,a)$$



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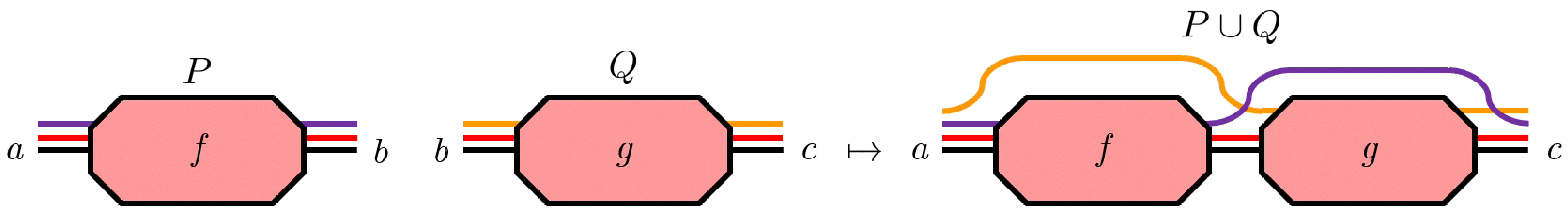
$$\mathsf{seq} : \mathbf{C}(a,b) \cup \mathbf{C}(b,c) \to \mathbf{C}(a,c)$$



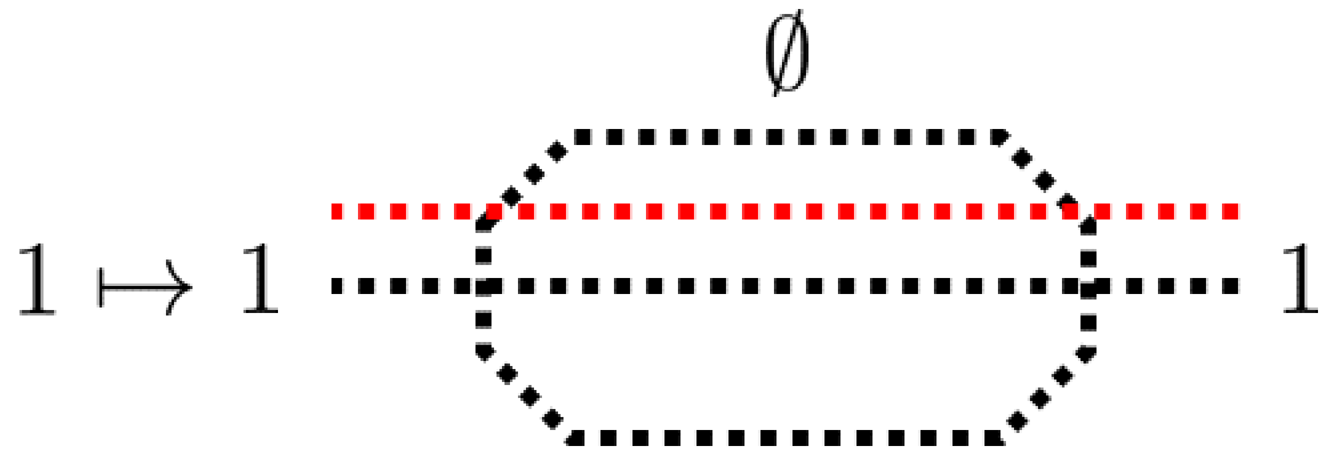
Compound State

$$R := \{\mathbb{B}, \mathbb{Z}, \ldots\} \qquad \mathcal{P}_f(R) = \{\emptyset, \{\mathbb{B}\}, \{\mathbb{Z}\}, \{\mathbb{B}, \mathbb{Z}\}, \ldots\}$$

$$\mathbf{V} = (\mathbf{Label}, \|, \mathsf{cst}_\emptyset, \cup, \mathsf{cst}_\emptyset) \qquad \mathbf{M} = (\mathbf{Set}, \times, 1)$$

$$\mathbf{C}(a,b) := \left(\sum_{Q \in \mathcal{P}_f(R)} \mathbf{Set}(\Pi_Q \times a, \Pi_Q \times b) \right) \xrightarrow{\ell} \mathcal{P}_f(R)$$

$$\mathsf{zero} \colon \mathsf{cst}_\emptyset \to \mathbf{C}(1,1)$$



Compound State

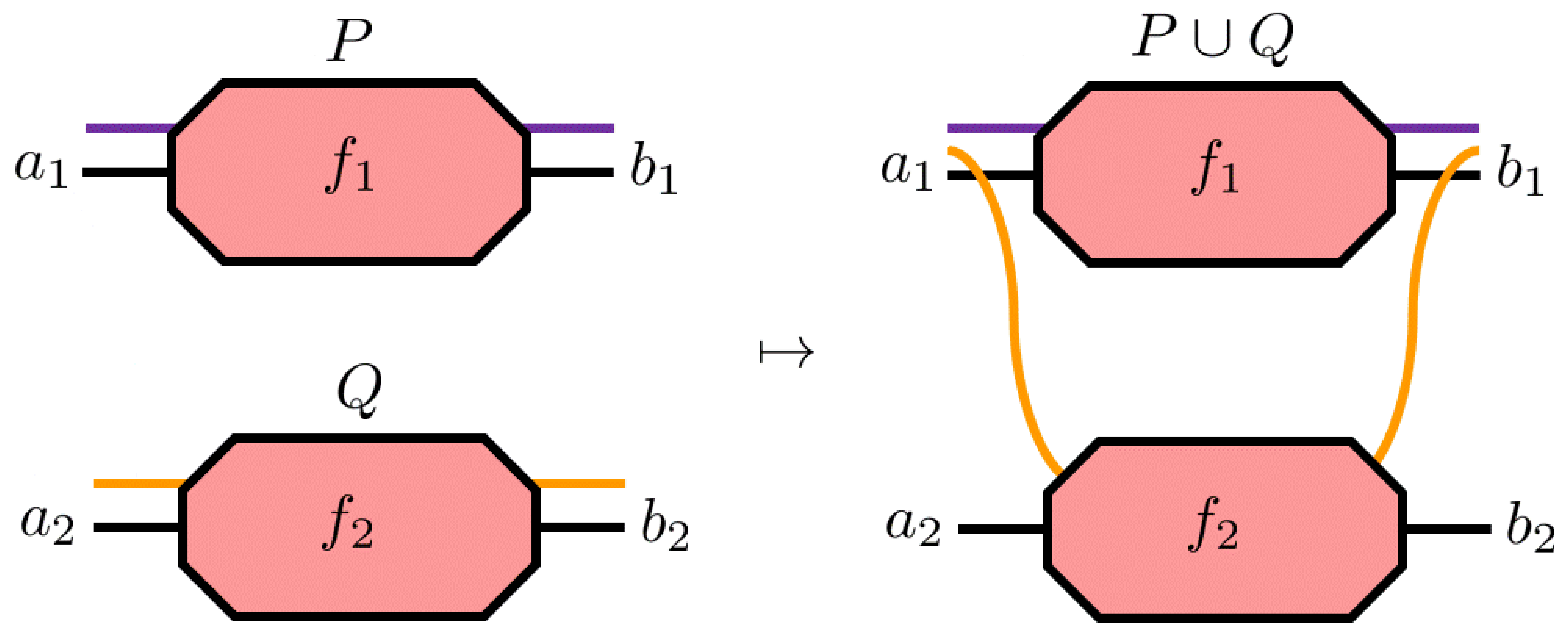
$R \coloneqq \{\mathbb{B}, \mathbb{Z}, \ldots\}$ $\quad$ $\mathcal{P}_f(R) = \{\emptyset, \{\mathbb{B}\}, \{\mathbb{Z}\}, \{\mathbb{B}, \mathbb{Z}\}, \ldots\}$

$$\mathbf{V} = (\mathbf{Label}, \|, \mathsf{cst}_\emptyset, \cup, \mathsf{cst}_\emptyset) \qquad \mathbf{M} = (\mathbf{Set}, \times, 1)$$

$$\mathbf{C}(a,b) \coloneqq \left(\sum_{Q \in \mathcal{P}_f(R)} \mathbf{Set}(\Pi_Q \times a, \Pi_Q \times b) \right) \xrightarrow{\ell} \mathcal{P}_f(R)$$

$$\mathsf{par}\colon \mathbf{C}(a_1, b_1) \| \mathbf{C}(a_2, b_2) \to \mathbf{C}(a_1 \times a_2, b_1 \times b_2)$$

$P \cap Q = \emptyset$



Compound State

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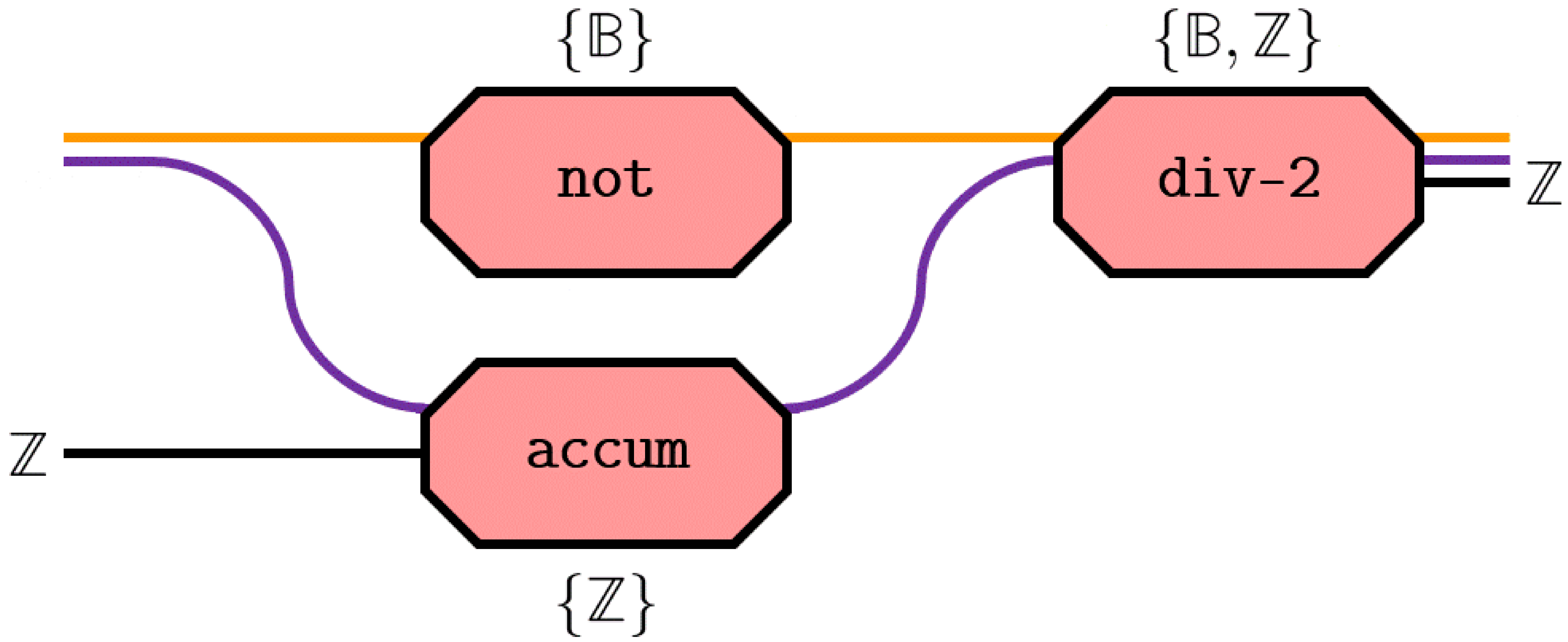


```
not : () -> ()
not(b, ()) = (¬b, ())

accum : ℤ -> ()
accum(s, y) = (y + s, ())

div-2 : () -> ℤ
div-2(b, s, ()) =
  if b then
    (b, s, ceil(s/2))
  else
    (b, s, floor(s/2))
```

Freyd and **Subset-Freyd**

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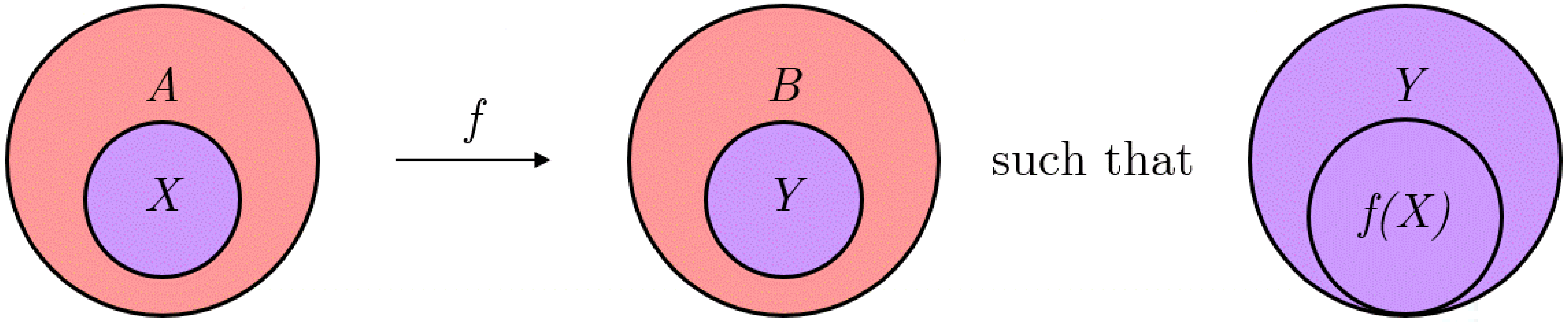
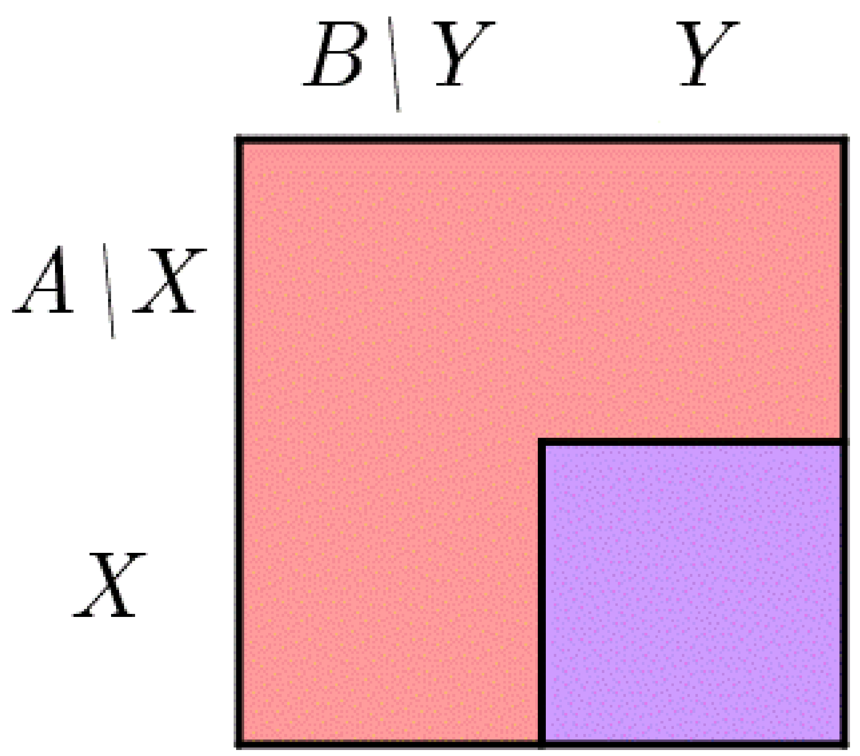
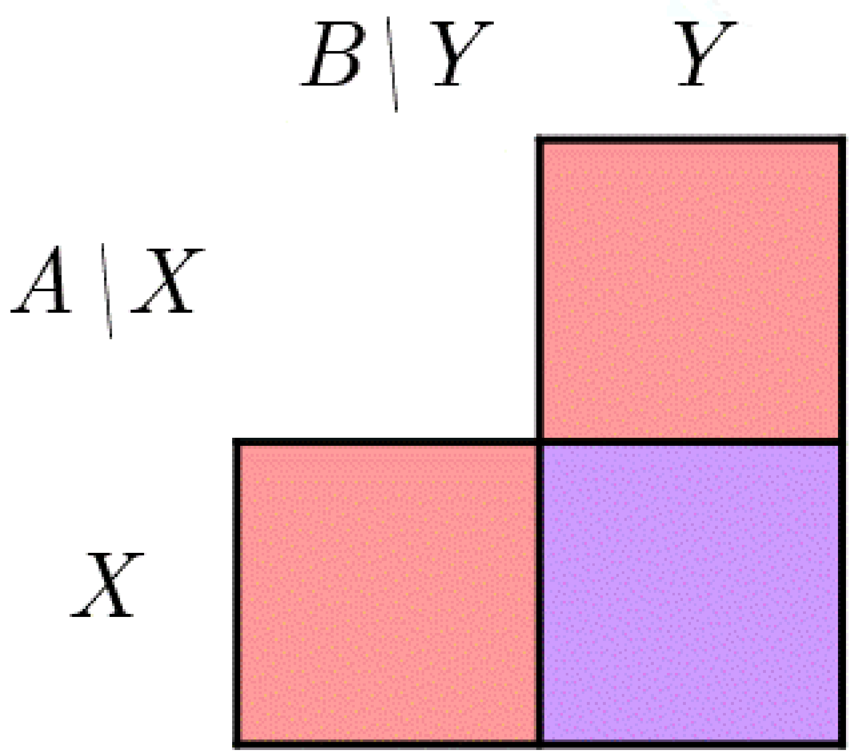




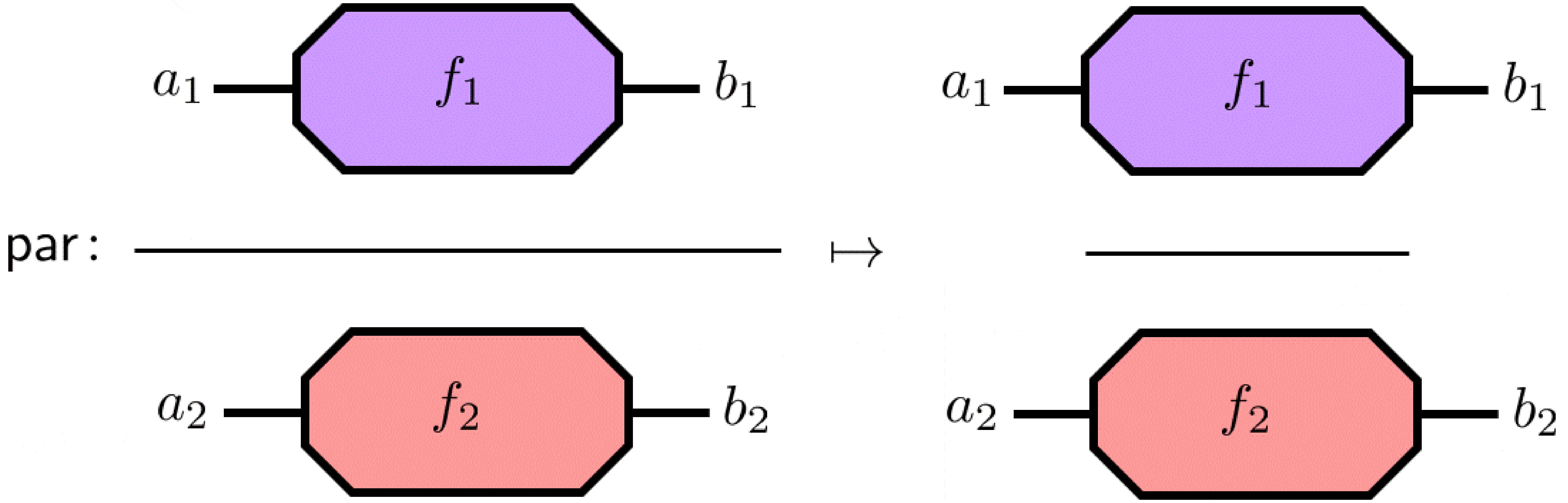


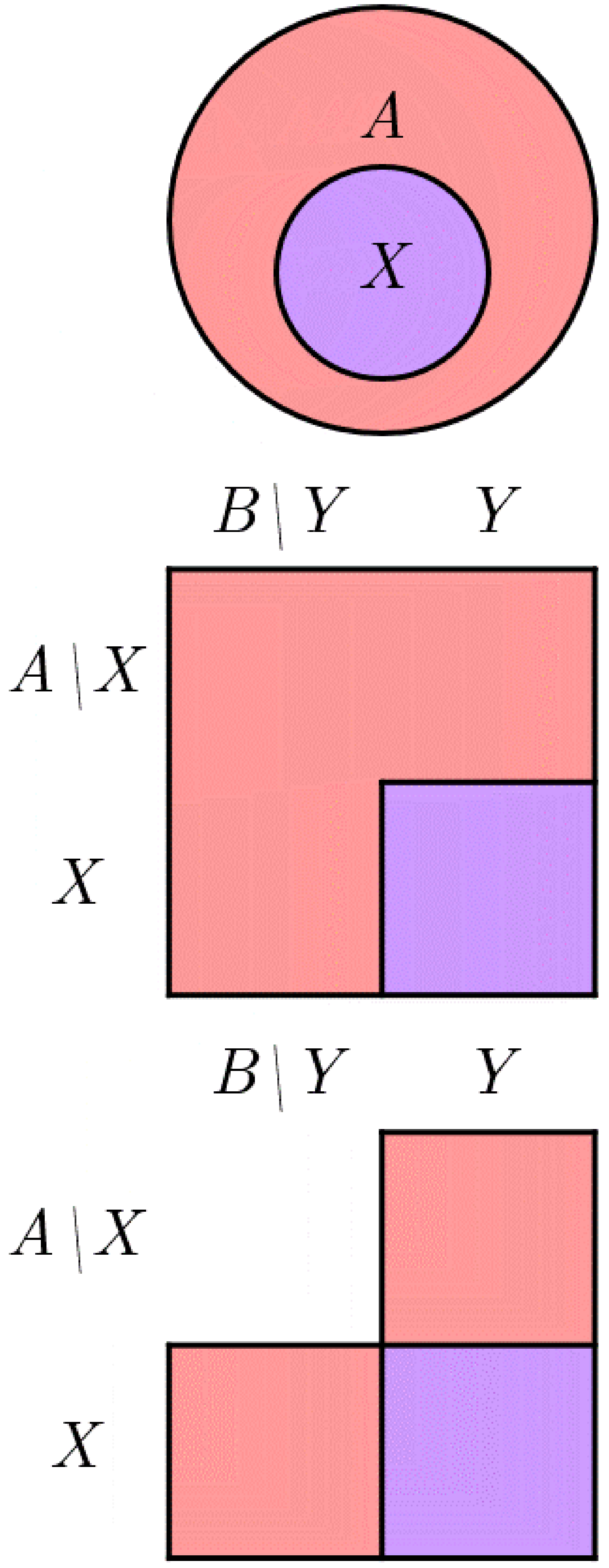
$$(X, A) \times (Y, B) := (X \times Y, A \times B)$$

$$(X, A) \otimes (Y, B) := \big(X \times Y, (A \times Y) \cup (X \times B)\big)$$

$$\big(\mathbf{Subset}, \otimes, (1, 1), \times, (1, 1)\big) \text{ is duoidal}$$

Freyd and **Subset-Freyd**





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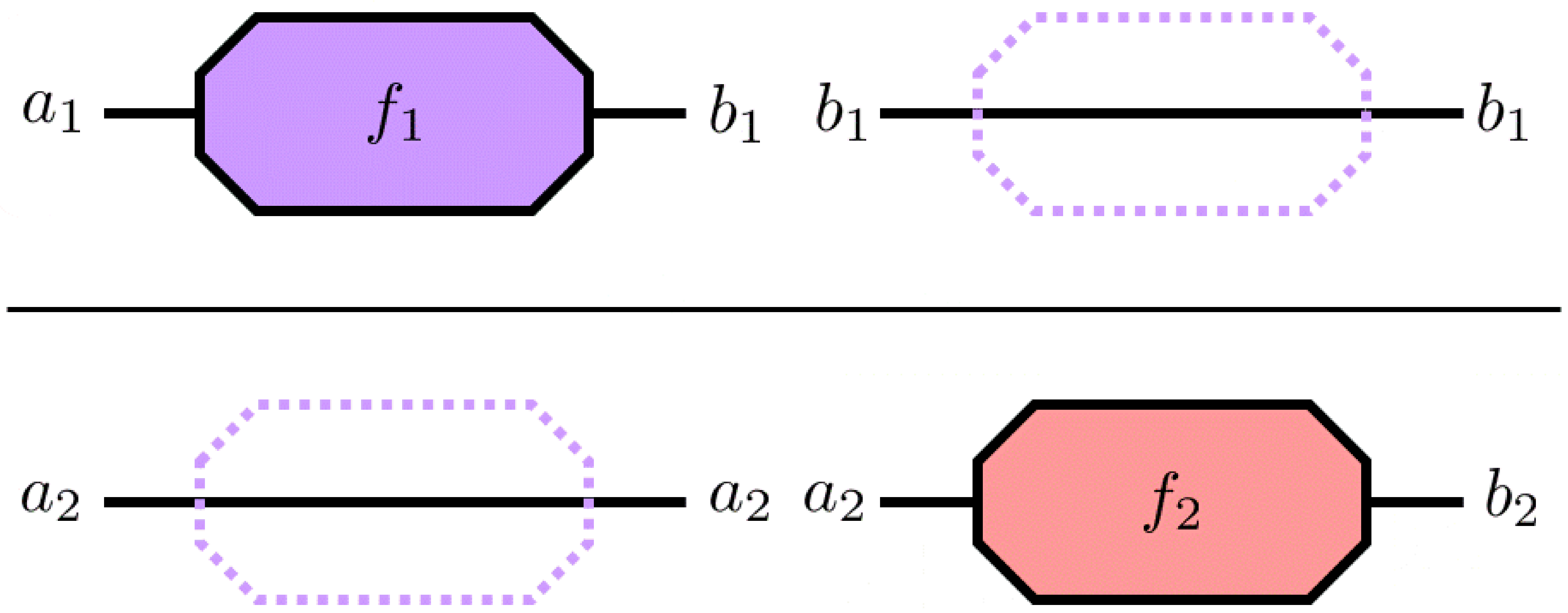
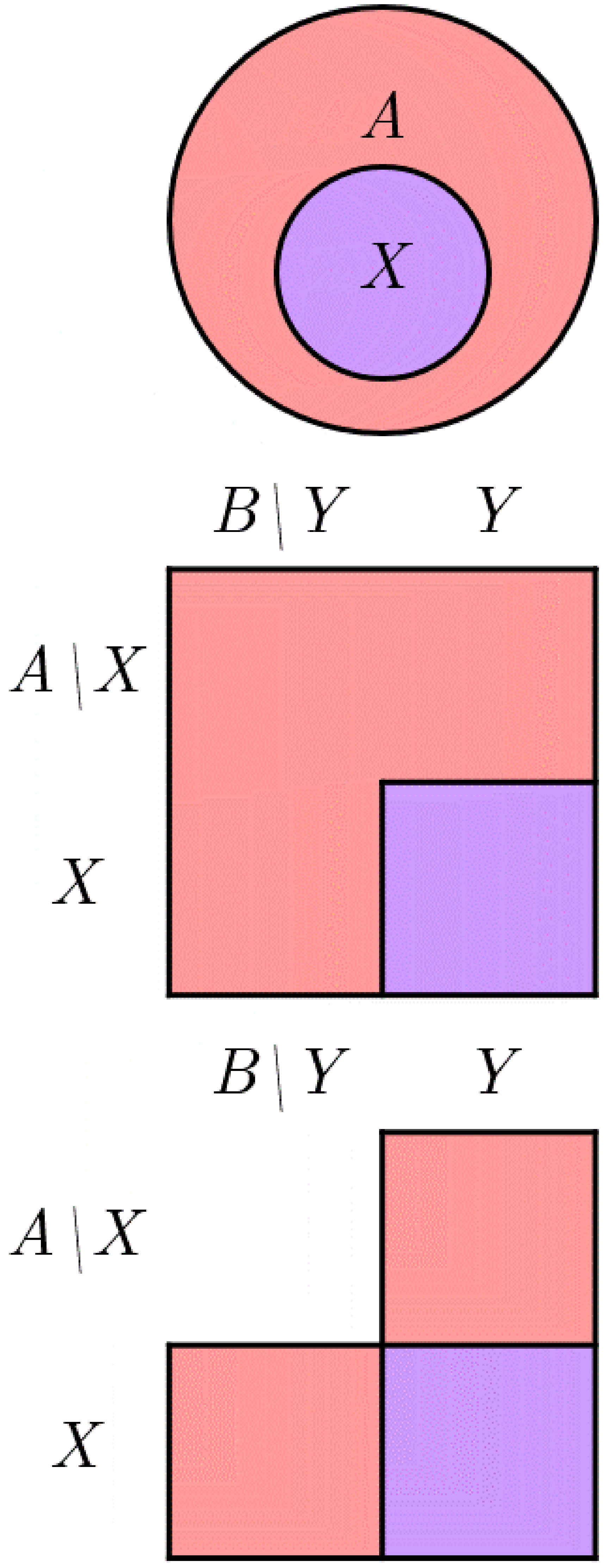
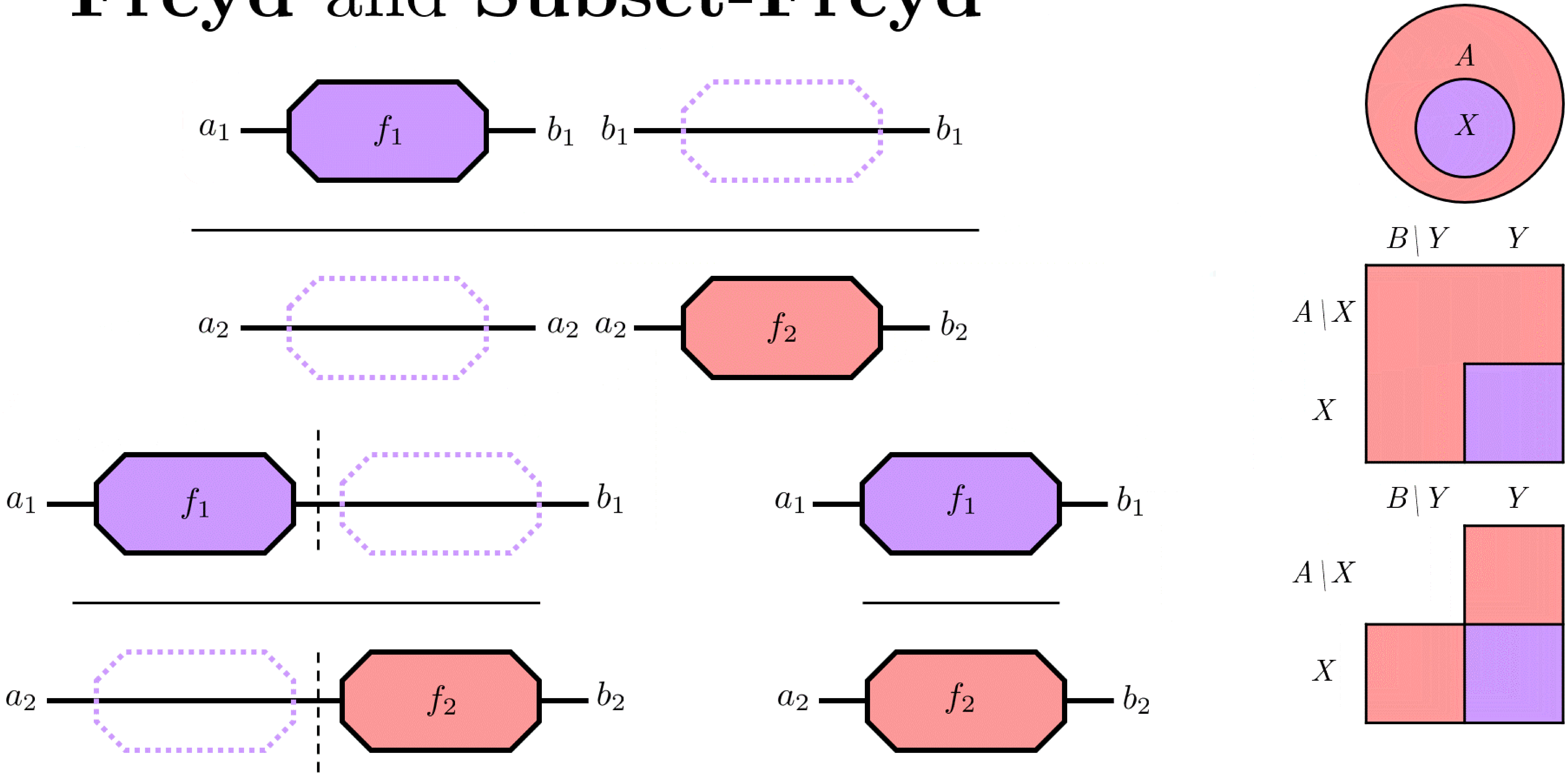




Freyd and Subset-Freyd



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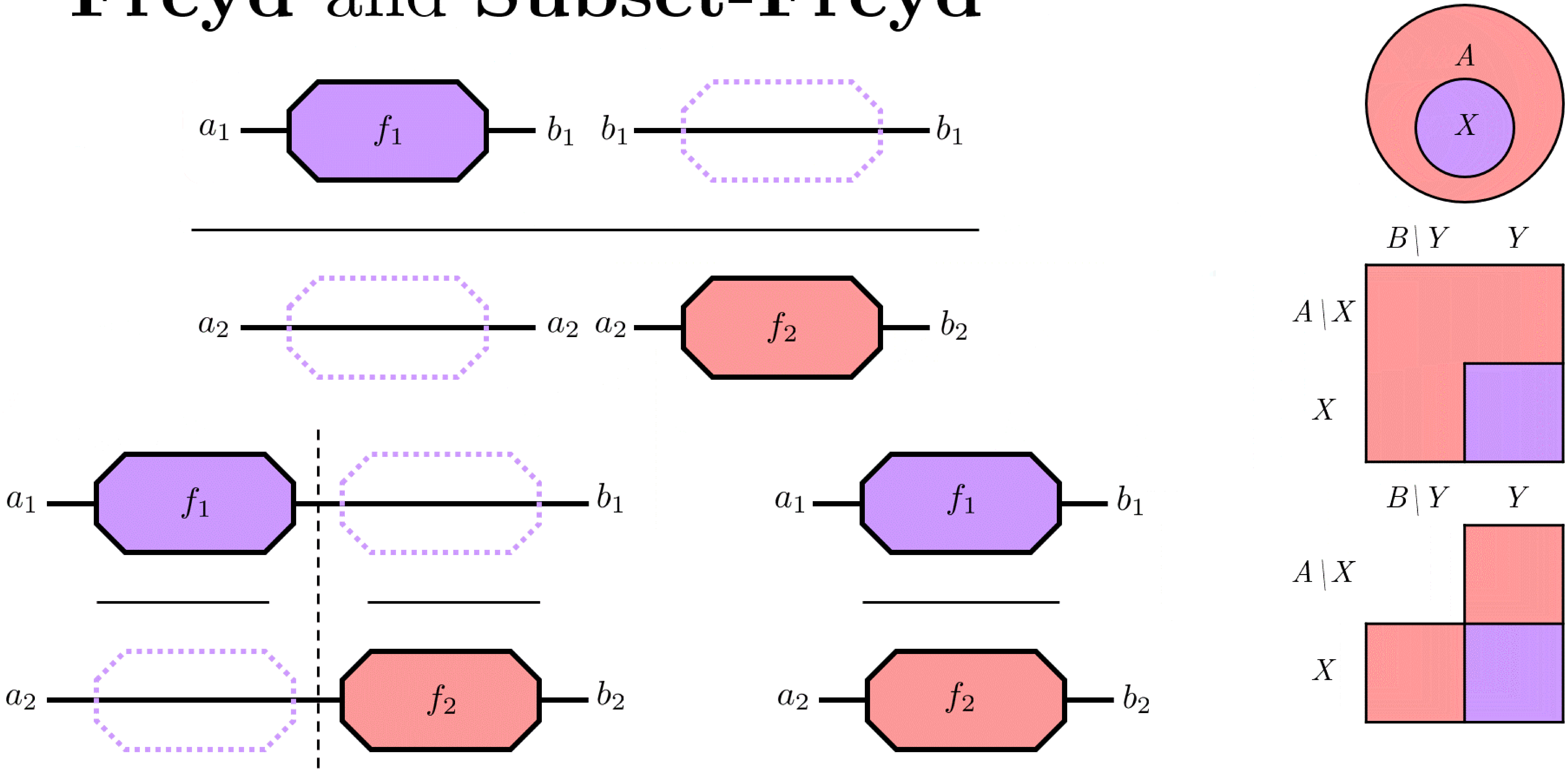

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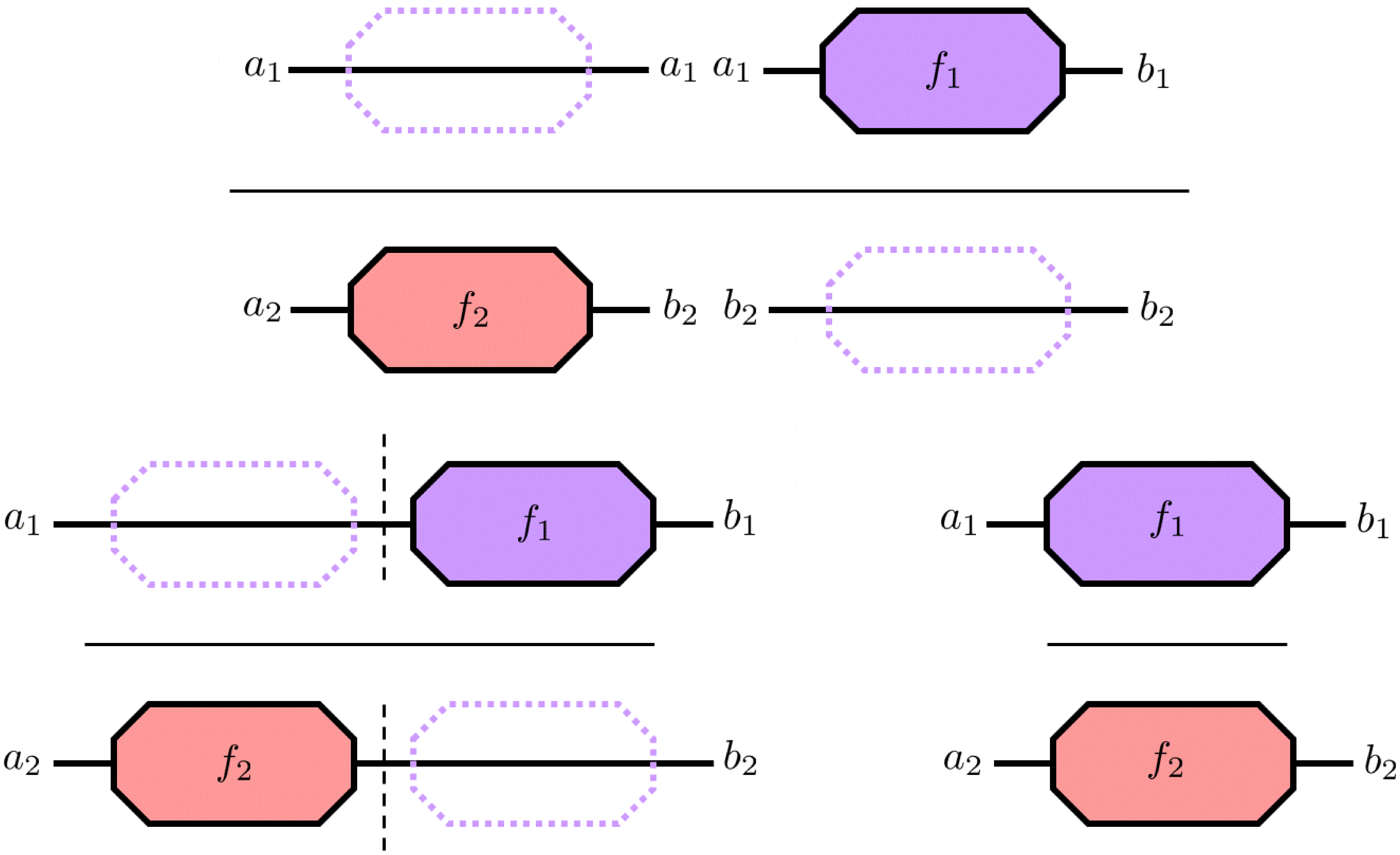

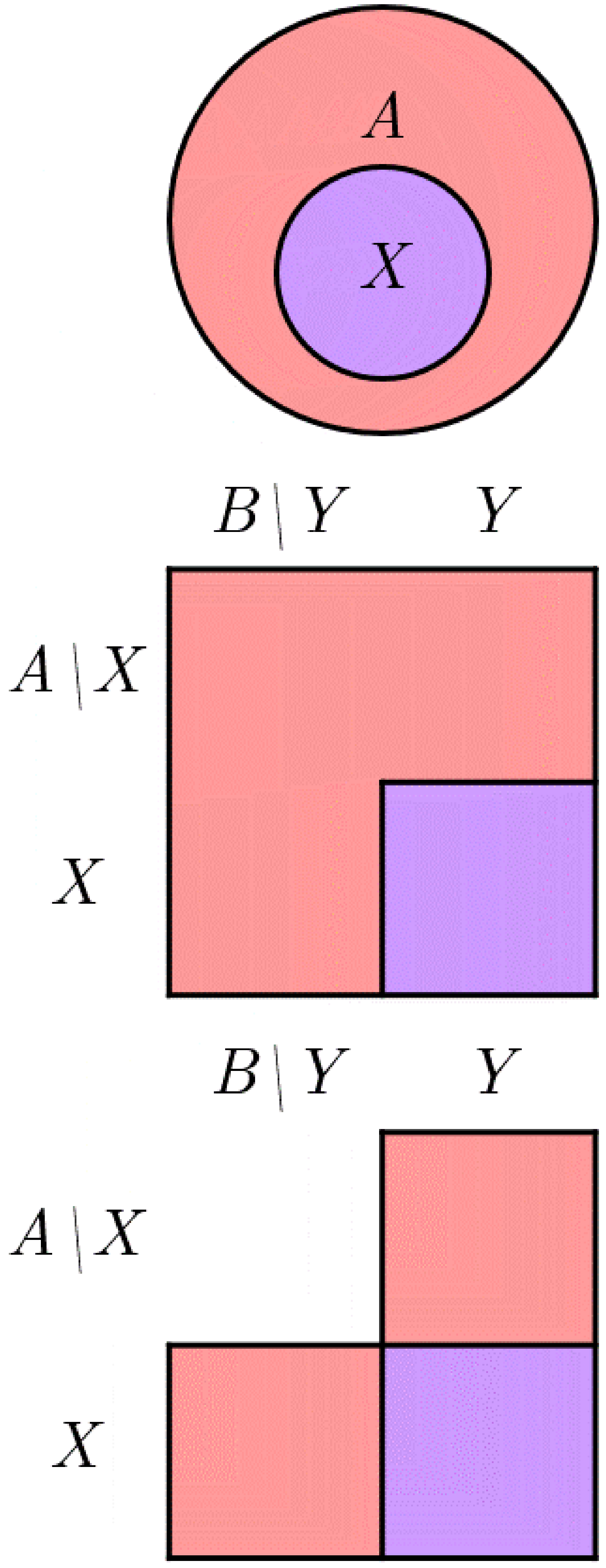

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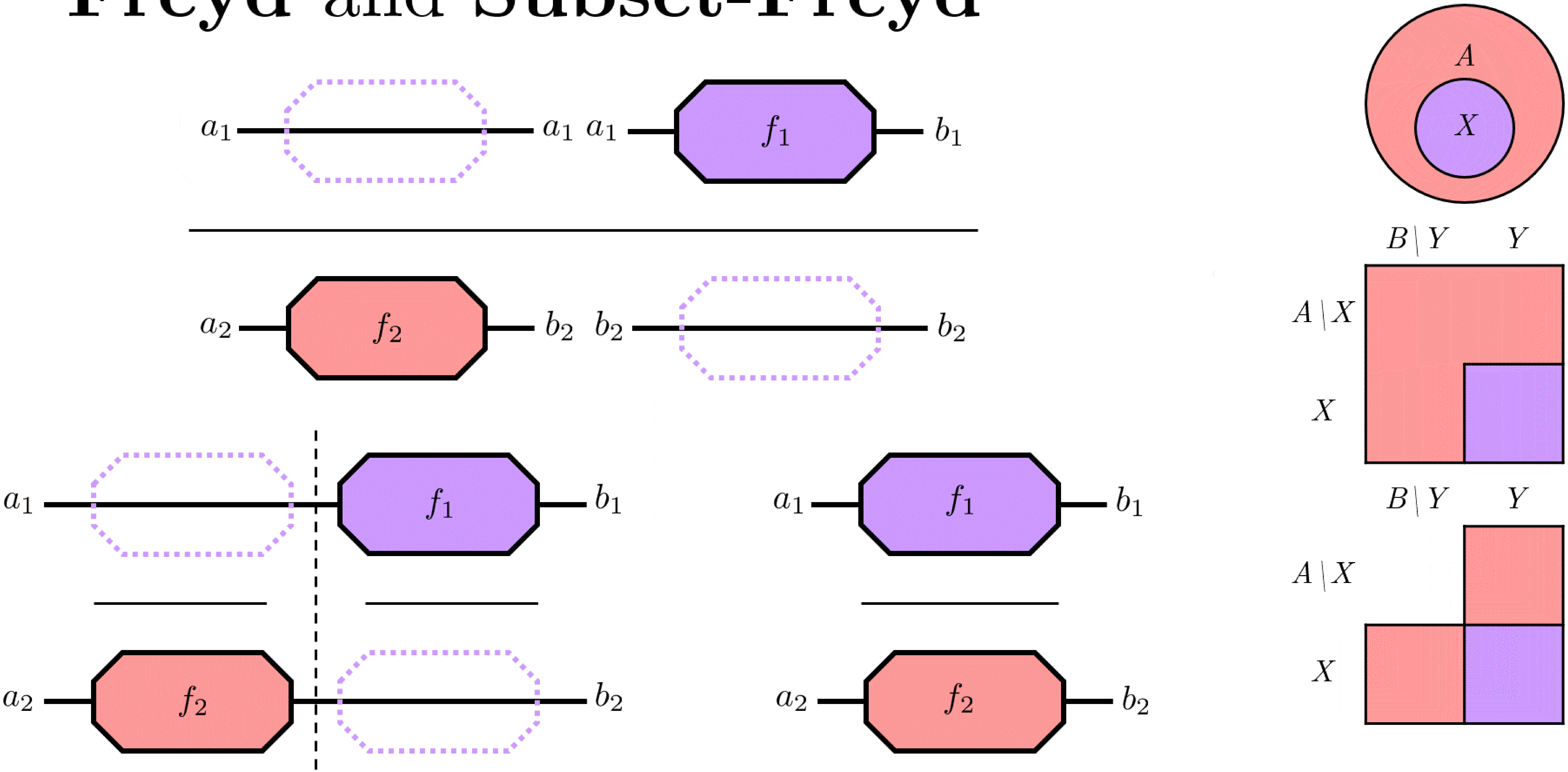

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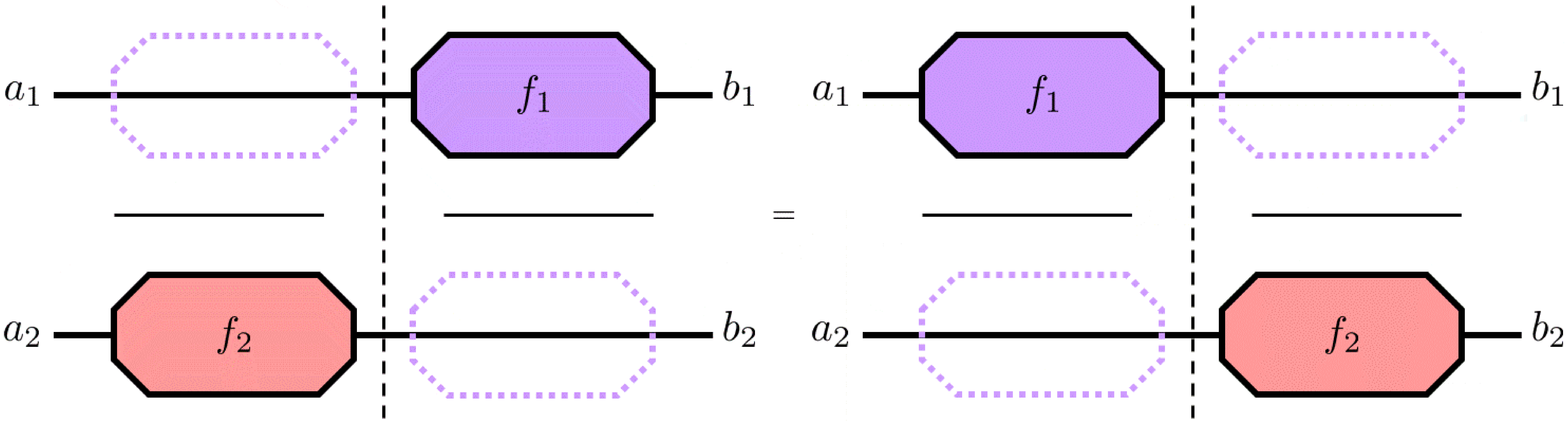
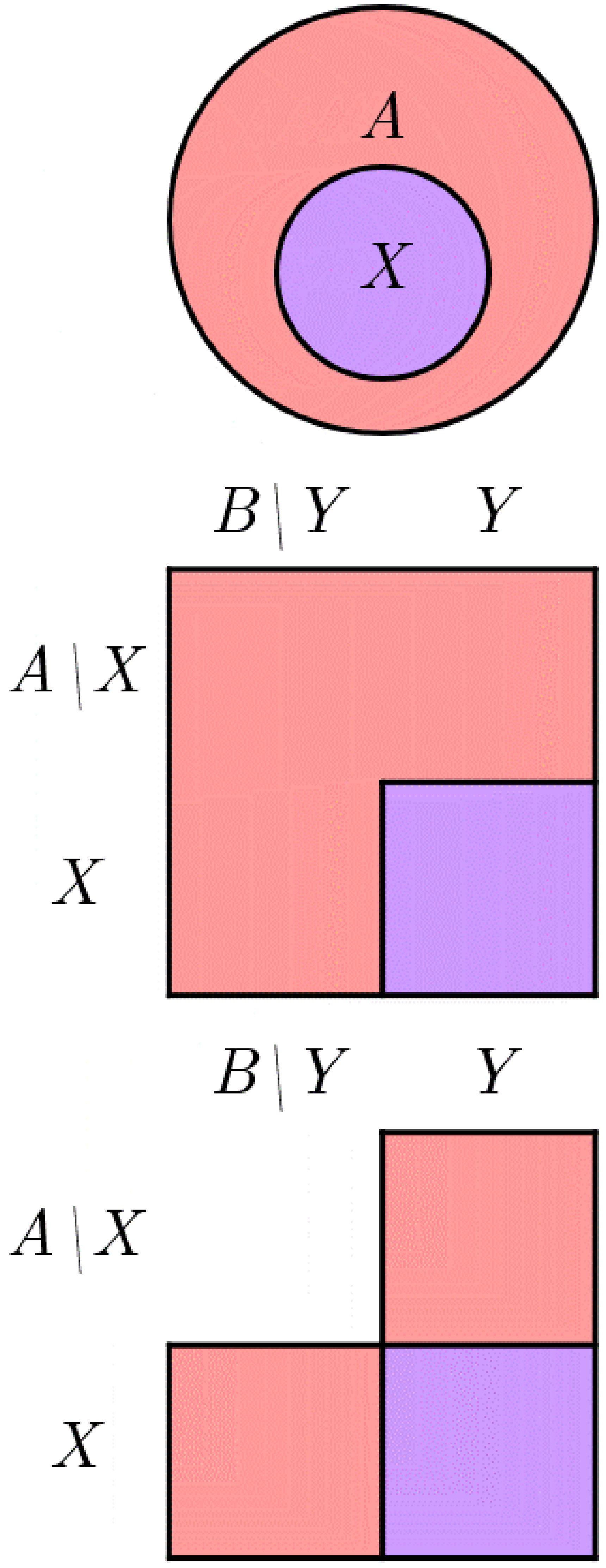
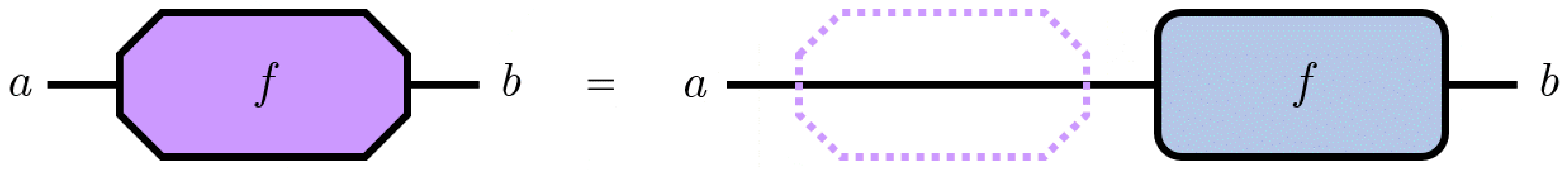


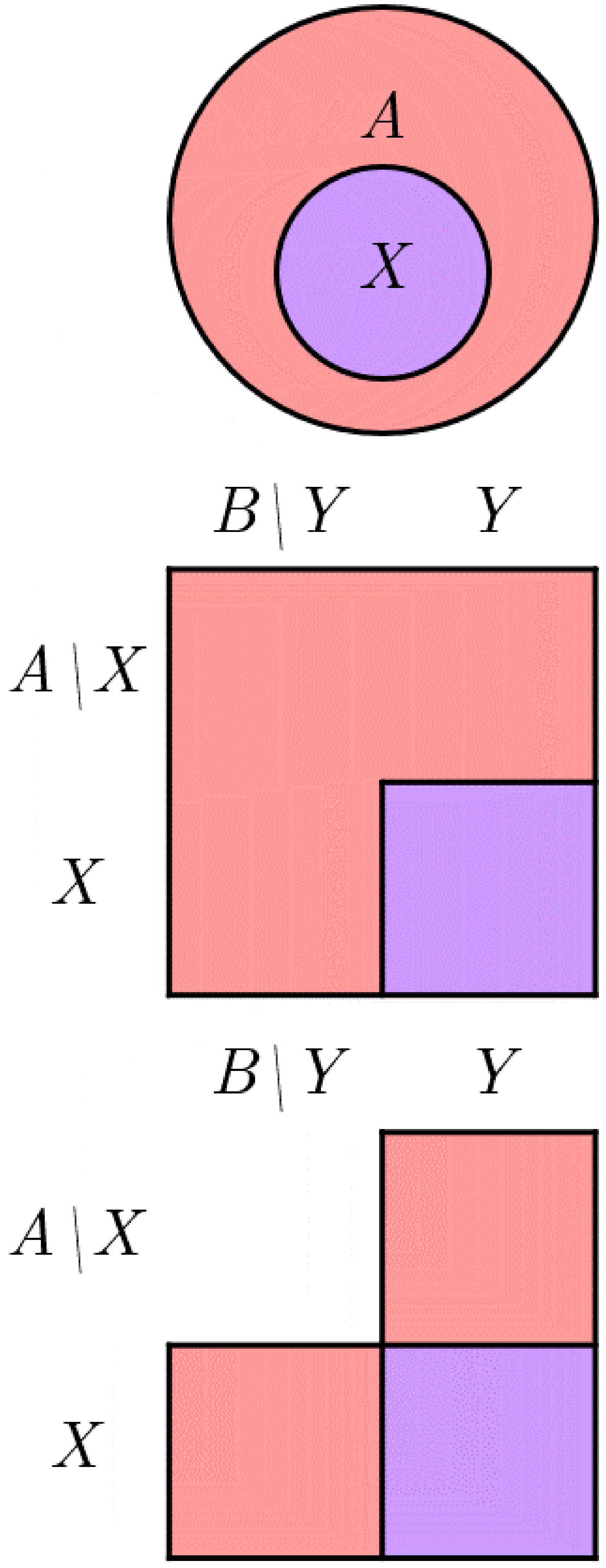


Freyd and **Subset-Freyd**

Freyd is equivalent to the full coreflective subcategory of **Subset-Freyd** s.t.



Subset-Freyd is Freyd with a little bit more information.



More in the paper

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- Abstract characterization of duoidally enriched Freyd categories as monoids in some category.

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- More examples, indexed state and Kleisli categories for changing Lawvere theories.

Conclusion

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Summary

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Thanks for listening!