Implication Semigroups of Binary Relations

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Roadmap

1. Implication algebras, Implication semigroups
2. Why these matter
3. Representations: what they are, why they matter
4. Defining relational representations
5. Decidability of Finite representation for **IA**
6. Undecidability for **ISG**

Implication Semigroups: What are they?

- There are several ways one can arrive at these.
- Take the standard signature of relation algebras $(L, \wedge, \vee, \neg, 0, 1, ;, 1', \smile)$, and restrict to the fragment $(\rightarrow, ;)$.
- Start with Abbot's implication algebras and then add a semigroup operation (;), which for our purposes, we interpret as relational composition.

Definition 1

An *implication algebra*[a] $\mathfrak{A}$ is a pair $(A, \to)$, with A a set and $\to$ a binary operation on A satisfying the following properties:

1. $(a \to b) \to a = a$ (Contraction)
2. $(a \to b) \to b = (b \to a) \to a$ (Quasi-commutativity)
3. $a \to (b \to c) = b \to (a \to c)$ (Exchange)

[a] Also known as Tarski algebras.

As the class of implication algebras is equationally definable, it forms a *variety*. We shall refer to this class as **IA**.

Implication Semigroups

Definition 2

An *implication semigroup* $\mathfrak{S}$ is a tuple $(S, \to, ;)$, with a carrier set S and $\to, ;$ binary operations on S where

1. $(S, \to)$ is an implication algebra
2. $(S, ;)$ is a semigroup
3. $((a \to b) \to b); c = (a; c \to b; c) \to b; c$ (Left quasi-additivity)
4. $c; ((a \to b) \to b) = (c; a \to c; b) \to c; b$ (Right quasi-additivity)

Similarly with **IA**, we will refer to the variety of Implication Semigroups as **ISG**.

Implication Semigroups: Why they matter

- The present work extends research on fragments of the full signature of relation algebras. Such fragments of relation algebras often are interesting on their own terms.
- It is not obvious which signatures will have decidable representation problems, for instance. (We define representation shortly.)
- We are particularly interested in understanding the effect on representability when moving to subsignatures of the standard presentation of a relation algebra.
- Implication algebras and their subsystems have been well-studied, including in substructural logic, and those sytems have a computational interpretation. Perhaps there's room for thinking about e.g. BCK logic with a semigroup operation.

Representations: What they are, why they matter

- A *representation* of an algebra $\mathfrak{A}$ is typically understood to mean a representation via some appropriate cannonical map f to some concrete algebra of sets, relations, etc.
- e.g. Stone's representation theorem [Sto36] for Boolean Algebras associates with every Boolean Algebra $\mathfrak{A}$ an algebra of clopen sets of ultrafilters.
- Representations became important for Relation Algebras because, as it turns out, the axioms of relation algebras have many different models, not all of which are isomorphic to a set relation algebra, or an algebra of binary relations using standard set-theoretically defined operations.
- Therefore, identifying conditions under which models are isomorphic (or not) to set a relation algebra became of interest; similarly, if representable as such an algebra of relations, identifying the conditions for a finite representation.

Relational Representation Defined

Definition 3

Let $\top \subseteq X \times X$ be a binary relation. Define $\mathfrak{A}(\top) = (\wp(\top), \to)$ where $\to$ is interpreted as proper Boolean implication defined below

$$a \to b = (\top \setminus a) \cup b$$

Definition 4

We say that $\mathfrak{A} \in \mathbf{IA}$ is *representable* if and only if it embeds into $\mathfrak{A}(\top)$ for some $\top \subseteq X \times X$. The embedding (usually denoted h) is called a *representation*. If $\mathfrak{A}$ embeds into $\mathfrak{A}(\top)$ and $\top$ is over a finite base X, then we say $\mathfrak{A}$ is also *finitely representable*.

Although $\top$ is conventionally an arbitrary maximal relation, this is not the only possible interpretation of the $\to$ operation for binary relations.

Definition 5

We say that the implication operator is *absolute* if we require $\top = X \times X$, else we say that it is *relative*.

Similarly to **IA** we also examine structures where the carrier set is a set of binary relations.

Definition 6

Let $\top \subseteq X \times X$ be a transitive binary relation. Define $\mathfrak{S}(\top) = (\wp(\top), \rightarrow, ;)$ where $\rightarrow$ is interpreted as proper Boolean implication and ; as proper relational composition defined as

$$a; b = \{(x, z) \mid \exists y \in X : (x, y) \in a, (y, z) \in b\}$$

Again checking $\mathfrak{S}(\top) \in$ **ISG** is relatively straightforward, note that they are closed under composition due to the transitivity of $\top$. Similarly to **IA**:

Definition 7

$\mathfrak{S} \in$ **ISG** is (finitely) representable if it embeds into $\mathfrak{S}(\top)$ for some transitive $\top$ (over a finite base).

Present work

- The finite representation problem for **IA** – we can read this as an algebra of unary or binary relations– is decidable.
- The (finite) representation decision problem for **ISG**, however, is undecidable.

Finite representation for **IA**, **ISG**

- The idea for **IA**: use Stone-style representation [Abo67, Die65, Ras74] to obtain a representation; this answers the general decidability question positively.
- The idea for **ISG**: use [HJ12] result on undecidability for Boolean monoids based on partial group embeddings and [Neu16] result that the representation problem for Boolean semigroups is undecidable to get, respectively, that the decision problem for absolute implication is undecidable and the decision problem for relative implication is also undecidable.

Finite representation for **IA**

Definition 8

The (finite) representation decision problem for implication algebras is a decision problem that takes an implication algebra with a (finite) carrier set as input. The algebra is a yes instance if and only if it is (finitely) representable.

Theorem 9 (Rasiowa 7.1)

For any implication algebra $\mathfrak{A} = (A, \to)$, there is a monomorphism h from $\mathfrak{A}$ to $(\wp(X), \to)$ of an arbitrary space X with $|X| \geq A$.

Rasiowa presents the preceding result for irreducible implicative filters [Ras74], but one can also state this using prime implicative filters, or maximal implicative filters.

Corollary 10

For any implication algebra $\mathfrak{A}$, if A is finite, then $\mathfrak{A}$ has a finite representation.

Corollary 11

$\mathbf{IA}$ *is finitely axiomatisable.*

Corollary 12

The (finite) representation problem for $\mathbf{IA}$ *is decidable.*

Undecidability of the finite representation problem **ISG**

The above theorem holds true for both interpretations of $\rightarrow$.

Theorem 13

The (finite) representation decision problem for is undecidable.

Proof Outline.

- The (finite) representation problem for Boolean Monoids and Boolean Semigroups is undecidable [Neu16, HHJ21]
- Prove that $(\rightarrow, ;)$-reduct of a Boolean Semigroup/Monoid is representable if and only if the said Boolean Semigroup is representable
- Conclude that the (finite) representation decision problem for is undecidable

Case 1: Relative $\to$

In a representation of the $(\to, ;)$-reduct of a Boolean Semigroup (with relative complementation) we have that

- The join is correctly represented because $(a \to b) \to b$ defines join
- The negation is correctly represented if 0 is represented as an empty relation because $a \to 0$ defines negation

Prove that if the $(\to, ;)$-reduct of a Boolean Semigroup is representable via some h then it is also representable via some h' that represents 0 as an empty relation. See details in the paper.

Case 2: Absolute $\rightarrow$

Take a Boolean Monoid (with absolute complementation) and its $(\rightarrow, ;)$-reduct.

1. Use the techniques from [Neu16] to represent the identity is as the diagonal (even if $1'$ is not in the signature)
2. 0 has to be represented as $\emptyset$ in the $(\rightarrow, ;)$-reduct:
   By contradiction, if $h(0)$ is not an empty relation, then all $(x, y) \in h(0)$ because anything composed with 0 equals 0. Because $0 \rightarrow a = 1$ for all $a \in A$ we also get that all pairs $(x, y) \in h(a)$. Thus the h is not faithful and not a representation.
3. Conclude undecidability by the argument on the previous slide.

Thank you!

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